AFFINE QUIVER SCHUR ALGEBRAS AND p-ADIC GL_n

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Abstract. In this paper we consider the (affine) Schur algebra introduced by Vignéras as the endomorphism algebra of certain permutation modules for the Iwahori-Matsumoto Hecke algebra. This algebra describes, for a general linear group over a p-adic field, a large part of the unipotent block over fields of characteristic different from p. We show that this Schur algebra is, after a suitable completion, isomorphic to the quiver Schur algebra attached to the cyclic quiver. The isomorphism is explicit, but nontrivial. As a consequence, the completed (affine) Schur algebra inherits a grading. As a byproduct we obtain a detailed description of the algebra with a basis adapted to the geometric basis of quiver Schur algebras. We illustrate the grading in the explicit example of $\mathrm{GL}_2(\mathbb{Q}_5)$ in characteristic 3.

Contents

1. Introduction

This paper deals with affine Schur algebras for p-adic groups over fields of characteristic different from p. Classical Schur algebras were introduced by Sandy Green [Gre80] as an algebraic tool to study polynomial representations of the general linear group GL_n over arbitrary fields and named after Schur because they arise as the endomorphism ring of the sum of certain permutation modules of the symmetric group S_n. Dipper and James [DJ89] introduced q-Schur algebras over arbitrary fields to study the modular representation theory of the finite

general linear groups $\mathrm{GL}_n(\mathbb{F}_q)$ in non-describing characteristic. The Schur algebras we consider in this paper are their analogues for the representation theory of the p-adic group $\mathrm{GL}_n(E)$, where E is a finite extension of $\mathbb{Q}_p$, over a field of characteristic different from p. As a main result, we show that (after a suitable completion) this algebraically defined algebra has a geometric realization as a convolution algebra with underlying vector space the equivariant cohomology of some partial quiver flag varieties introduced in [SW11] under the name *quiver Schur algebras*.

Let $\Bbbk$ be an algebraically closed field of characteristic $\ell \neq p$, such that the cardinality of the residue field of E is not congruent to 1 modulo ℓ. We are interested in the category of smooth representations of $\mathrm{G} = \mathrm{GL}_n(E)$ over the field $\Bbbk$ (or equivalently the category of nondegenerate representations of the global Hecke algebra of locally constant compactly supported functions on G). This is known to have a block decomposition by inertial classes of supercuspidal support [B84], [Vig98], [SS14]. In this article, we are interested in the so-called *unipotent block* $\mathcal{B}$ which contains the trivial representation.

As in the case of $\mathrm{GL}_n(\mathbb{F}_q)$, the Schur algebra will not describe the whole unipotent block, but rather some subcategory $\mathcal{B}^1$ which is the lowest layer in a finite filtration $\mathcal{B}^1 \subseteq \mathcal{B}^2 \subseteq \mathcal{B}^3 \subseteq \ldots \subseteq \mathcal{B}$. Namely, let $I \subset \mathrm{G}$ be an Iwahori subgroup and let $\mathcal{I}$ be the annihilator of the G-representation $\Bbbk[I\backslash \mathrm{G}]$ (inside the global Hecke algebra). Then $\mathcal{B}^i \subset \mathcal{B}$ is the full subcategory consisting of all representations annihilated by $\mathcal{I}^i$. The categories $\mathcal{B}^i$ are abelian. It is proved in [Vig03] that the first layer $\mathcal{B}^1$ is equivalent to the category of all modules for the *affine Schur algebra* $\mathcal{S}$,

$$\mathcal{B}^1 \cong \mathcal{S} - \mathrm{Mod}, \tag{1.1}$$

where $\mathcal{S}$ is defined as the endomorphism ring

$$\mathcal{S} \;=\; \mathrm{End}_{\Bbbk[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]}\left(\bigoplus_{J\subseteq\mathbb{I}} \Bbbk[\mathrm{P}^J\backslash\mathrm{G}/\mathrm{I}]\right) \;=\; \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{J\subseteq\mathbb{I}} \mathbf{v}_J\mathcal{H}\right).$$

Here $\mathcal{H} = \Bbbk[I\backslash \mathrm{G}/I]$ is the (affine) *Iwahori-Matsumoto Hecke algebra*, [IM65], and the sum is taken over all standard parahoric subgroups P^J attached to a subset J of the set $\mathbb{I}$ of (finite) simple reflections, and $\mathbf{v}_J\mathcal{H}$ is the corresponding trivial representation induced to $\mathcal{H}$. In particular, $\mathcal{S}$ contains $\mathcal{H}$ as an idempotent subalgebra from setting $J = \emptyset$. Note however that $\mathcal{B}^1$, alias $\mathcal{S} - \mathrm{Mod}$, is in general not equivalent to $\mathcal{H} - \mathrm{Mod}$, since $\mathcal{B}^1$ contains in addition the cuspidal representations, which are not included in the subcategory $\mathcal{H}-\mathrm{Mod}$ of $\mathcal{S}-\mathrm{Mod}$.

We expect that $\mathcal{B}$ in fact only differs from $\mathcal{B}^1$ by self-extensions depending on the cuspidal support of the corresponding simple modules, and thus $\mathcal{B}^1$ contains quite detailed information about the unipotent block $\mathcal{B}$.

Note that the classification of irreducible representations in $\mathcal{B}^1$ (or equivalently in the unipotent block $\mathcal{B}$) is provided by [Vig98], [MS14], and a convenient

labelling set for the irreducible modules is given by certain multisegments, extending the Bernstein-Zelevinsky classification of irreducible modules for the Iwahori-Matsumoto Hecke algebras, [BZ76], [Zel80] in characteristic zero. The block decomposition and classification in [Vig98] is via the local Langlands correspondence for GL_n in characteristic $\ell \neq p$, that is an extension of the local Langlands correspondence over the complex numbers, [HT01], [Har08], [Hen00], [Scho13] (or [Wed08] for an overview). In particular, this gives the rank of the Grothendieck group of $\mathcal{B}^1$.

In this article, we take this one step further by providing tools for a better understanding of extensions between simple modules and moreover of the structure and the homological properties of the categories involved, as well as making a connection with geometry. To do so, we compare the affine Schur algebras to the *quiver Schur algebras* from [SW11] attached to the cyclic quiver with e vertices (viewed as the oriented affine Dynkin diagram for $\hat{\mathfrak{sl}}_e$). These algebras contain the so-called *quiver Hecke algebras* or *KLR*-algebras, originally introduced in [KL09], [Rou08], see also [VV11]. Over $\Bbbk = \mathbb{C}$, their graded module categories furthermore provide by [SW11] a categorification of the generic Hall algebra (in the sense of [Schi12]) for the cyclic quiver with e vertices. Hereby e is the multiplicative order modulo ℓ of the cardinality of the residue field of E and $e = \infty$ if $\ell = 0$.

Given a fixed dimension vector $\mathbf{d}$ for the cyclic quiver on e vertices, one considers the space of *flagged nilpotent representations* with dimension vector $\mathbf{d}$, that is, representations together with a filtration such that the associated graded is semisimple. In contrast to the KLR-algebras we allow arbitrary partial flags instead of full flags only. Fixing a sequence $\hat{\boldsymbol{\lambda}}$ of dimension vectors for the successive quotients we denote this space $\mathcal{Q}(\hat{\boldsymbol{\lambda}})$. Following the ideas of Chriss and Ginzburg [CG10] we consider the "Steinberg type" variety $\mathcal{Z}(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}}) = \mathcal{Q}(\hat{\boldsymbol{\lambda}}) \times_{\mathrm{Rep}_\mathbf{d}} \mathcal{Q}(\hat{\boldsymbol{\mu}})$. The quiver Schur algebra $\mathbf{A_d}$ is then its $\mathrm{GL}_\mathbf{d}(\mathbb{C})$-equivariant Borel-Moore homology

$$\mathbf{A_d} \quad = \quad \bigoplus_{(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}})} H^{BM}_{\mathrm{GL}_\mathbf{d}}(\mathcal{Z}(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\lambda}})),$$

equipped with the convolution product. By construction, this algebra comes along with *a $\mathbb{Z}$-grading* and with a faithful representation, see [SW11].

Crucial for us here is that via the faithful representation we see that the quiver Schur algebra can be defined over any field, in particular over the field $\Bbbk$.

Main result: Our main result (Theorem 9.7) is that the affine Schur algebra and the quiver Schur algebra (both over $\Bbbk$) are isomorphic as algebras after suitable completions. More precisely, we construct a sequence of isomorphisms of algebras

$$\widehat{\mathcal{S}}_\mathbf{i} \xrightarrow{\text{Proposition 9.1}} \widehat{\mathbf{C}}_\mathbf{i} \xrightarrow{\text{Proposition 9.6}} \widehat{\mathbf{B}}_\mathbf{i} \xrightarrow{\text{Proposition 9.4}} \widehat{\mathbf{A}}_\mathbf{i},$$

where the intermediate algebras are certain twisted versions of quiver Schur algebras. This, in particular, implies that the category of representations M in $\mathcal{B}^1$ with fixed generalized central character (in the sense that each element in M is annihilated by some power of this central character) inherits a grading. This category, in particular, includes blocks of finite-length representations.

The existence of such a $\mathbb{Z}$-grading seems to be quite unexpected and has no explanation in the p-adic representation theory at the moment. Although the modules are of infinite length, the graded pieces are finite dimensional and so the grading allows us to consider Jordan-Hölder multiplicities degree-wise where they then, in fact, become finite and well-defined. Hence we can use formal power series to express the graded multiplicities.

In small examples, our isomorphism allows us to give a *complete* and *explicit* description of this category in terms of the path algebra of a quiver with generators and relations, an example is given in Section 10. In particular, it allows us to compute extensions between simple modules in small examples. This provides a first step towards general results about the homological algebra of $\mathcal{B}^1$, based on results on quiver Hecke algebras.

The proof of the main result relies on a very careful comparison of faithful representations of all involved algebras. The final result is then an *explicit* (non-trivial) isomorphism.

Besides the main theorem, the paper contains some fundamental results about the algebras involved. For instance, we construct several generating sets for the affine Schur algebras (see in particular Corollary 4.13 and Proposition 6.19), explicit faithful representations (in Section 4.1) and geometrically adapted bases (in Section 4.4). The paper also contains (see Section 8.3) explicit formulae for Demazure (divided difference) operators interacting with multiplication by polynomials, which we believe should play an important role in a possible categorification result. They generalise crucial formulae from the categorification of quantum groups, see e.g. [KL10], [KLMS12], and well-known formulae from the geometry of flag varieties.

In characteristic zero and for generic q, the affine q-Schur algebra was studied in detail by Richard Green [Gre99] who also realised it as a quotient of the quantum group for $\hat{\mathfrak{gl}}_n$. In this case a complete presentation of the algebra is available, [DG07]. In our more general situation such a presentation does not exist yet, but our faithful representations turn the problem of finding such a presentation into a problem of linear algebra. Moreover our explicit formulae should make it possible to generalise the geometric results for quiver Schur algebras defined over $\mathbb{k} = \mathbb{C}$ to the positive characteristic case with q a root of unity.

We have tried to make this paper as self-contained as possible, in order to make it accessible to readers both from a representation theoretic or a number theoretic background.

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2. Preliminaries

We fix a prime p and a natural number $n \geq 2$ and consider the general linear group $\mathrm{G} = \mathrm{GL}_n(E)$ for a finite extension E of $\mathbb{Q}_p$, the field of p-adic numbers. The field E has a local ring $\mathfrak{o}$ of integers, whose quotient by its maximal ideal $\mathfrak{p}$ is a finite field of characteristic p. We let q denote the cardinality of this residue field. We furthermore fix an algebraically closed field $\Bbbk$ of characteristic $\ell \geq 0$, $\ell \neq p$ and let e be the multiplicative order of q in $\Bbbk$. We assume $q \neq 1 \mod \ell$.

2.1. **The extended affine Weyl group.** We start by recalling the definition and basic facts of the extended affine Weyl group attached to G. For more details see e.g. [IM65] or [Gre02], [Lus83] for a description in terms of periodic permutations.

The *extended affine Weyl group* associated to G is the group W generated by a set $\mathbb{I}_0 = \{s_0, \dots, s_{n-1}\}$ of *simple reflections* of order two and an element τ of infinite order, given by the following presentation:

$$\mathrm{W} \quad = \quad \left\langle \tau, s_i, 0 \leq i \leq n-1 \;\middle|\; \begin{array}{l} s_{\overline{i}}^2 = 1, \quad \tau s_{\overline{i}} = s_{\overline{i-1}}\tau \\ s_{\overline{i+1}} s_{\overline{i}} s_{\overline{i+1}} = s_{\overline{i}} s_{\overline{i+1}} s_{\overline{i}} \end{array} \right\rangle. \tag{2.1}$$

where $\overline{i} \in \{0, \dots, n-1\}$ with $\overline{i} \equiv i \mod n$.

Using the relation $\tau s_{\overline{i}} = s_{\overline{i-1}}\tau$ we can (in a unique way) write every element $w \in W$ as $x\tau^j$ for some x contained in the subgroup generated by I_0 and $j \in \mathbb{Z}$. Define the length of $w = x\tau^j$ as $\ell(w) = \ell(x)$, where $\ell(x) = r$ with r minimal such that $x = s_{i_1} \cdots s_{i_r}$ for some $i_j \in 0, \dots, n-1$.

We view W as a subgroup of G by choosing lifts of its elements as follows: For $i = 1, \dots, n-1$, we choose the corresponding permutation matrix interchanging the ith and $(i+1)$st rows and columns. For s_0 we take the matrix with entry 1 in position (j,j) for $j = 2, \dots, n-1$, the uniformizer ϖ (a fixed generator of $\mathfrak{p}$) in position $(n,1)$ and its inverse ϖ^{-1} in position $(1,n)$, and all other entries being zero. Finally τ has ϖ in position $(n,1)$ and 1 in positions $(j,j+1)$ for $j = 1, \dots, n-1$, with again all other entries being zero.

There is another presentation of W as semi-direct product $\mathfrak{S} \ltimes \mathfrak{X}$, where $\mathfrak{S}$ is the symmetric subgroup generated by $\mathbb{I} = \{s_1, \dots, s_{n-1}\}$ and $\mathfrak{X}$ is a free abelian (multiplicative) group generated by $X_1, \dots, X_n$, on which $\mathfrak{S}$ acts by permuting the generators. More specifically, a general element of $\mathfrak{X}$ is a Laurent monomial $X_1^{a_1} \cdots X_n^{a_n}$ with $a_i \in \mathbb{Z}$ and $s_i X_i s_i = X_{i+1}$. A representative of X_i in G can

be chosen to be a matrix with 1's along the diagonal, except in position (i,i) where we put the uniformizer ϖ.

Lemma 2.1. *An isomorphism of groups* $\mathrm{W} \cong \mathfrak{S} \rtimes \mathfrak{X}$ *is given by*

$$\begin{aligned} s_i &\mapsto s_i \quad (i = 1, \dots, n-1), \\ s_0 &\mapsto s_{n-1} \cdots s_2 s_1 s_2 \dots s_{n-1} X_1 X_n^{-1}, \\ \tau &\mapsto s_{n-1} \cdots s_1 X_1. \end{aligned} \tag{2.2}$$

Its inverse sends X_1 *to* $s_1 \cdots s_{n-1}\tau$ *and of course* s_i *to* s_i *for* $1 \leq i \leq n-1$.

Proof. A direct computation shows that the assignments define group homomorphisms which are mutually inverse. □

From now on we will identify the two presentations (so that for instance the equality (*) in the next formula makes sense).

Let $\flat$ be the automorphism of W fixing generators $s_1, \dots, s_n$ and sending X_i to X_i^{-1} for $i = 1, \dots, n$. For convenience, we record that

$$\begin{aligned} s_0^\flat &= s_{n-1} \cdots s_1 \cdots s_{n-1} X_1^{-1} X_n \overset{(*)}{=} s_{n-1} \cdots s_1 \cdots s_{n-1} s_0 s_{n-1} \cdots s_1 \cdots s_{n-1} \\ \tau^\flat &= s_{n-1} \cdots s_1 X_1^{-1} = s_{n-1} \cdots s_1 \tau^{-1} s_{n-1} \cdots s_1. \end{aligned}$$

Lemma 2.2. *Let* $wp \in \mathrm{W}$ *with* $w \in \mathfrak{S}$ *and* $p = X_1^{a_1} \cdots X_n^{a_n} \in \mathfrak{X}$.

i.) If $a_i \geq 0$ *for all* $i \in \{1, \dots, n\}$, *then* wp *can be expressed in terms of generators from* $\mathbb{I}$ *and* τ *(involving only positive powers of* τ*).*

ii.) If $a_i \leq 0$ *for all* $i \in \{1, \dots, n\}$, *then* wp *can be expressed in terms of generators from* $\mathbb{I}$ *and* τ^{-1} *(involving only negative powers of* τ*).*

Proof. This follows directly from the fact that $X_i = s_{i+1} \cdots s_{n-1} \tau s_1 \dots s_i$. □

2.2. **Parabolic subgroups and shortest double coset representatives.** For a subset $J \subsetneq \mathbb{I}_0$, we denote by $\mathrm{W}_J = \langle s_i \mid s_i \in J \rangle$ the parabolic subgroup generated by J. Note that this is a finite group, isomorphic to the direct product of some symmetric groups.

Let now $J, K \subseteq \mathbb{I}$. Then each double coset in $\mathrm{W}_K \backslash \mathrm{W} / \mathrm{W}_J$ contains a unique shortest (i.e. minimal length) coset representative. We denote the set of shortest double coset representatives by $D_{K,J}$. If moreover J_1, J_2 are both subsets of $K \subseteq \mathbb{I}$, we denote the (finite) set of shortest coset representatives in $\mathrm{W}_{J_1} \backslash \mathrm{W}_K / \mathrm{W}_{J_2}$ by $D^K_{J_1,J_2}$.

For $d \in D_{K,J}$, the set $dJ \cap K$ is defined as the intersection of K with all elements in W of the form ds_jd^{-1} for $s_j \in J$, i.e. $dJ \cap K = \{s \in K \mid d^{-1}sd \in J\}$. Moreover we abbreviate $dJ = dJ \cap \mathbb{I}$.

For $d \in D_{K,J}$ and any element w in $\mathrm{W}_K d \mathrm{W}_J$, there exist unique elements $w_K \in \mathrm{W}_K, w_J \in \mathrm{W}_J$, and $a \in D^J_{d^{-1}K \cap J, \emptyset}$, respectively $b \in D^K_{\emptyset, dJ \cap K}$ such that

$$w = w_k da = bdw_J \quad \text{with} \quad l(w) = l(w_k)l(d)l(a) = l(b)l(d)l(w_J). \tag{2.3}$$

2.3. **Another set of double coset representatives.** Let $J \subseteq \mathbb{I}$. A monomial $X_1^{a_1} \cdots X_n^{a_n} \in \mathfrak{X}$ is called *J-dominant* if $a_i \geq a_j$ for all $i \leq j$ such that s_i and s_j are conjugate in W_J. It is called *J-antidominant* if $a_i \leq a_j$ for all $i \leq j$ as above. We denote by $\overrightarrow{\mathfrak{X}_J}$ and $\overleftarrow{\mathfrak{X}_J}$ the set of J-dominant and the set of J-antidominant elements respectively. Note that each W_J-orbit in $\mathfrak{X}$ contains a unique J-dominant element and a unique J-antidominant element.

Proposition 2.3. *Let $J, K \subset \mathbb{I}$. The sets*

$$\begin{aligned} \Delta_{K,J} &= \left\{ dp \mid d \in D^{\mathbb{I}}_{K,J}, p \in \overrightarrow{\mathfrak{X}}_{d^{-1}K\cap J} \right\} \quad \text{and} \\ \nabla_{K,J} &= \left\{ dp \mid d \in D^{\mathbb{I}}_{K,J}, p \in \overleftarrow{\mathfrak{X}}_{d^{-1}K\cap J} \right\} \end{aligned}$$

both form a complete set of inequivalent coset representatives in $\mathrm{W}_K \backslash \mathrm{W} / \mathrm{W}_J$.

Proof. We only prove the first claim, since the second one is analogous. We first show that every double coset in $\mathrm{W}_K \backslash \mathrm{W} / \mathrm{W}_J$ contains an element from $\Delta_{K,J}$. We know it contains an element from $D_{K,J}$, so let $y \in D_{K,J}$ and write $y = wf$ for $w \in \mathfrak{S}, f \in \mathfrak{X}$. Using (2.3), we can find $d \in D^{\mathbb{I}}_{K,J}, w_K \in \mathrm{W}_K, a \in D^J_{d^{-1}K\cap J,\emptyset}$ such that $w = w_K da$. Then $y = w_K d(afa^{-1})a$. Now let $t \in \mathrm{W}_{d^{-1}K\cap J}$ such that $p = tafa^{-1}t^{-1} \in \mathfrak{X}$ is J-dominant. Then $dtd^{-1} \in \mathrm{W}_{K\cap dJ} \subseteq \mathrm{W}_K$ and therefore $y = w_K dtd^{-1}dpa \in \mathrm{W}_K dp \mathrm{W}_J$ with $d(afa^{-1}) \in \Delta_{K,J}$ as claimed.

Conversely, we need to show that the element dp is the unique element in $\mathrm{W}_K dp \mathrm{W}_J$ with $d \in D^{\mathbb{I}}_{K,J}$ and $p \in \overrightarrow{\mathfrak{X}}_{d^{-1}K\cap J}$, so take an element $w_1 dpw_2 \in \mathrm{W}_K dp \mathrm{W}_J$ and write it as $w_1 dpw_2 = \sigma f$ with $\sigma \in \mathfrak{S}, f \in \mathfrak{X}$. Note that necessarily $\sigma = w_1 dw_2$ and $f = w_2^{-1} pw_2$. Assume $\sigma \in D^{\mathbb{I}}_{K,J}$ and $f \in \overrightarrow{\mathfrak{X}}_{d^{-1}K\cap J}$. Then $w_1 dw_2 = d$, so writing $w_1 = ab$ with $a \in D^K_{\emptyset,dJ\cap K}, b \in \mathrm{W}_{dJ\cap K}$ and thus $d = w_1 dw_2 = ad(d^{-1}bd)w_2$ with $(d^{-1}bd)w_2 \in \mathrm{W}_J$, we have a presentation of d of the form given in (2.3), from which we deduce $a = 1$ and $(d^{-1}bd)w_2 = 1$, in particular $w_2 \in \mathrm{W}_{d^{-1}K\cap J}$. Hence $y = dw_2^{-1}pw_2$ with $w_2 \in \mathrm{W}_{d^{-1}K\cap J}$. Since p is the unique $d^{-1}K \cap J$-dominant element in its $\mathrm{W}_{d^{-1}K\cap J}$-orbit, it follows that $w_2^{-1}pw_2 = p$ and hence $y = dp$. $\square$

3. The Hecke algebra and Hecke modules

The goal of this section is to define the Iwahori-Matsumoto-Hecke algebra, originally introduced in [IM65], and to construct a faithful representation. Most of the statements can be found in [Lus89]. We collect some basic facts and give detailed proofs for those for which we could not find an appropriate reference.

3.1. **The Iwahori-Matsumoto Hecke algebra of** G**.** We start with the following presentation of the Hecke algebra due to Bernstein:

Definition 3.1. The *Iwahori-Matsumoto Hecke algebra* associated with G is the unitary $\Bbbk$-algebra $\mathcal{H} = \mathcal{H}_n$ generated by $T_1, \ldots, T_{n-1}$, and $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$, subject to the defining relations

$$\begin{array}{llll}
\text{(H-1)} & (T_i - q)(T_i + 1) = 0, & & \text{for } 1 \leq i \leq n-1, \\
\text{(H-2)} & T_i T_j = T_j T_i & \text{if } |i-j| > 1, & \text{for } 1 \leq i, j \leq n-1, \\
\text{(H-3)} & T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & & \text{for } 1 \leq i \leq n-2, \\
\text{(H-4)} & X_i X_i^{-1} = 1 = X_i^{-1} X_i & & \text{for } 1 \leq i \leq n-1, \\
\text{(H-5)} & X_i X_j = X_j X_i & & \text{for } 1 \leq i \leq n-1, \\
\text{(H-6)} & T_i X_j = X_j T_i & \text{if } |i-j| > 1, & \text{for } 1 \leq i, j \leq n-1, \\
\text{(H-7)} & T_i X_i T_i = q X_{i+1} & & \text{for } 1 \leq i \leq n-2,
\end{array}$$

where q is the cardinality of the residue field of E.

Note that in particular, for $i = 1, \ldots, n-1$,

$$(T_i - q)T_i = -(T_i - q) \quad \text{and} \quad (T_i + 1)T_i = q(T_i + 1). \tag{3.1}$$

Moreover, the T_i are invertible with

$$T_i^{-1} = q^{-1}T_i + (q^{-1} - 1) \quad \text{and} \quad T_i^2 = (q-1)T_i + q. \tag{3.2}$$

The following two sets are $\Bbbk$-bases of $\mathcal{H}$, [Lus89, Proposition 3.7]:

$$\{X_1^{a_1} \cdots X_n^{a_n} T_w \mid w \in \mathfrak{S}, a_i \in \mathbb{Z}\}, \quad \{T_w X_1^{a_1} \cdots X_n^{a_n} \mid w \in \mathfrak{S}, a_i \in \mathbb{Z}\} \tag{3.3}$$

We denote by $\mathcal{P} = \mathcal{P}_n = \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ the subalgebra of $\mathcal{H}$ generated by the $X_i^{\pm 1}$, where $i = 1, \ldots, n$. Note that the subalgebra in $\mathcal{P}$ given by symmetric (Laurent) polynomials is central by (H-7).

The original definition of the Iwahori-Matsumoto Hecke algebra of G over the field $\Bbbk$ is the convolution algebra $\Bbbk[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ of compactly supported I-bi-invariant functions on G with values in $\Bbbk$, where I is the Iwahori subgroup. Any such function can be written as $\sum_{w \in W} a_w \chi_{\mathrm{I}w\mathrm{I}}$ with some $a_w \in \Bbbk$ where $\chi_{\mathrm{I}w\mathrm{I}}$ is the characteristic function on the double coset $\mathrm{I}w\mathrm{I}$.

Abbreviating $T_w = \chi_{\mathrm{I}w\mathrm{I}}$ gives us the following presentation, [IM65, Theorem 3.5], of $\Bbbk[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$: The algebra is generated by T_{s_i} for $i = 0, \ldots n-1$ and T_τ subject to relations (using notation as in (2.1)):

$$T_{s_i}^2 = (q-1)T_{s_i} + q, \quad T_{s_{\overline{i+1}}} T_{s_{\overline{i}}} T_{s_{\overline{i+1}}} = T_{s_{\overline{i}}} T_{s_{\overline{i+1}}} T_{s_{\overline{i}}}, \quad T_\tau T_{s_{\overline{i}}} = T_{s_{\overline{i-1}}} T_\tau \tag{3.4}$$

The following isomorphism between $\Bbbk[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ and the Bernstein presentation justifies the twofold use of the same notation:

Lemma 3.2. *We have an isomorphism of $\Bbbk$-algebras*

$$\begin{array}{rccl}
\Theta: & \Bbbk[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}] & \to & \mathcal{H} \\
& T_{s_i} & \mapsto & T_i, \quad (i = 1, \ldots, n-1), \\
& T_{s_0} & \mapsto & q^{n-1} X_1^{-1} X_n (T_{n-1} \cdots T_2 T_1 T_2 \cdots T_{n-1})^{-1}, \\
& T_\tau & \mapsto & q^{-(n-1)/2} T_{n-1} \cdots T_1 X_1.
\end{array}$$

This isomorphism sends T_τ^n to $X_1 \cdots X_n$.

Proof. By definition the map preserves the first two relations in (3.4). For the last one note that the assignment for T_0 is precisely such that this relation holds for $i = 1$. Abbreviating $z = T_{n-1} \cdots T_1$ and using the relation $T_{i-1}^{-1} T_i T_{i-1} = T_i T_{i-1} T_i^{-1}$ for $2 \leq i \leq n-1$ we deduce $T_{i-1}^{-1} z X_1 T_i = z T_i^{-1} X_1 T_i$. Hence it remains only to check one case of the last relation namely that $T_\tau T_{s_0} T_\tau^{-1}$ maps to $T_{s_{n-1}}$ which is done by computing the image of $T_\tau^2 T_{s_1} T_\tau^{-2}$ and using the fact that

$$\begin{aligned}
(T_{n-1} \cdots T_1 X_1)^2 &= T_{n-1} T_{n-2} T_{n-1} T_{n-3} T_{n-2} T_{n-4} T_{n-3} \ldots T_2 T_3 T_1 T_2 X_1 T_1 X_1 \\
&= T_{n-2} T_{n-1} T_{n-3} T_{n-2} T_{n-4} T_{n-3} \ldots T_2 T_3 T_1 T_2 T_1 X_1 T_1 X_1 \\
&= q T_{n-2} T_{n-1} T_{n-3} T_{n-2} T_{n-4} T_{n-3} \ldots T_2 T_3 T_1 T_2 X_1 X_2,
\end{aligned}$$

which, setting $T_{n-2} T_{n-1} T_{n-3} T_{n-2} T_{n-4} T_{n-3} \ldots T_2 T_3 T_1 T_2 = h$, then gives

$$\begin{aligned}
(T_{n-1} \cdots T_1 X_1)^2 T_1 (T_{n-1} \cdots T_1 X_1)^{-2} &= h X_1 X_2 T_1 (X_1 X_2)^{-1} h^{-1} \\
&= h T_1 h^{-1} = T_{n-1} h h^{-1} = T_{n-1}.
\end{aligned}$$

Hence we have a well-defined surjective algebra homomorphism. To see that it is an isomorphism, recall that the T_w, $w \in \mathrm{W} \cong \mathfrak{S} \rtimes \mathfrak{X}$ form a basis of $\Bbbk[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$, and consider the (second) basis of $\mathcal{H}$ from (3.3). Then comparing Θ with Lemma 2.1 directly shows that Θ identifies the two bases. □

From now on we will freely identify the two presentations.

3.2. **The ideals $\mathbf{v}_J\mathcal{H}$.** For $J \subseteq \mathbb{I}$ with corresponding parabolic subgroup W_J of W we denote by $\mathcal{H}_J \subset \mathcal{H}$ the (finite-dimensional) Hecke subalgebra generated by $\{T_i \mid s_i \in J\}$ and define

$$\mathbf{v}_J \;=\; \sum_{w \in W_J} T_w, \quad \text{and} \quad \overline{\mathbf{v}}_J = \sum_{w \in W_J} (-q)^{-l(w)} T_w. \tag{3.5}$$

We often abbreviate $\mathbf{v} = \mathbf{v}_{\mathbb{I}}$ and $\overline{\mathbf{v}} = \overline{\mathbf{v}}_{\mathbb{I}}$. Note that $\Bbbk\mathbf{v}_J$ and $\Bbbk\overline{\mathbf{v}}_J$ are the 1-dimensional trivial respectively sign (right) $\mathcal{H}_J$-modules via (3.1). They generate the following right ideals in $\mathcal{H}$ which play the role of permutation modules in the representation theory of the symmetric group.

Lemma 3.3. *The right ideals*

$$\begin{aligned}
\mathcal{H}^J_{\mathrm{triv}} &= \{h \in \mathcal{H} \mid (T_i - q)h = 0 \text{ for all } i \text{ such that } s_i \in J\}, \\
\mathcal{H}^J_{\mathrm{sgn}} &= \{h \in \mathcal{H} \mid (T_i + 1)h = 0 \text{ for all } i \text{ such that } s_i \in J\}
\end{aligned}$$

are principal right ideals in $\mathcal{H}$, generated by $\mathbf{v}_J$ respectively $\overline{\mathbf{v}}_J$.

Proof. Clearly, $\mathbf{v}_J$ is contained in $\mathcal{H}^J_{\mathrm{triv}}$. Now $\mathcal{H}$ is a by (3.3) a free left module over $\mathcal{H}_J$. Therefore, we obtain

$$\mathcal{H}^J_{\mathrm{triv}} = \{h \in \mathcal{H}_J \otimes_{\mathcal{H}_J} \mathcal{H} \mid (T_i - q)h = 0 \text{ for all } i \text{ such that } s_i \in J\} = \mathbf{v}_J \mathcal{H}$$

and the first claim follows. The second is similar. □

Corollary 3.4. *In case $J \subseteq \mathbb{I}$, a $\Bbbk$-basis for $\mathbf{v}_J\mathcal{H}$ respectively $\overline{\mathbf{v}}_J\mathcal{H}$ is given by*

$$\left\{T_w X_1^{a_1} \cdots X_n^{a_n} \,\middle|\, w \in D^{\mathbb{I}}_{J,\emptyset}, a_i \in \mathbb{Z}\right\}.$$

Proof. This follows directly from Lemma 3.3 and (3.3). □

Note that the ideals $\mathbf{v}_J\mathcal{H}$ and $\overline{\mathbf{v}}_J\mathcal{H}$ have isomorphic endomorphism rings. To pass between them we will later need the algebra automorphism $\sharp$ of $\mathcal{H}$, which is the q-analogue of $\flat$, defined on the generators by

$$\begin{array}{lllr} T_i & \mapsto & T_i^\sharp = q - 1 - T_i = -qT_i^{-1}, & (i = 1, \dots, n-1), \\ X_j & \mapsto & X_j^\sharp = X_j^{-1} & (j = 1, \dots, n). \end{array} \tag{3.6}$$

Remark 3.5. If $f \in \mathcal{P}$ is s_i-invariant, then $fT_i^\sharp = T_i^\sharp f$ for $i = 1, \dots, n-1$.

3.3. **A completion of $\mathcal{H}$.** Recall, [Lus89, Proposition 3.11], that the centre of $\mathcal{H}$ is given by

$$Z(\mathcal{H}) \quad = \quad \Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathfrak{S}}. \tag{3.7}$$

For $\mathbf{a} = (a_1, \dots, a_n) \in (\Bbbk^*)^n$ define the corresponding central character

$$\chi_{\mathbf{a}}: \qquad Z(\mathcal{H}) \to \Bbbk, \quad \text{by restriction from} \quad X_i \mapsto a_i.$$

Two characters $\chi_{\mathbf{a}}$ and $\chi_{\mathbf{a}'}$ coincide if and only if $\mathbf{a}$ and $\mathbf{a}'$ belong to the same $\mathfrak{S}$-orbit. We can decompose any finite-dimensional representation M of $\mathcal{H}$ as $M = \bigoplus_\chi M_\chi$, where χ runs over a set of representatives of $\mathfrak{S}$ -orbits on $\Bbbk^n$ and M_χ consists of all elements of M which are annihilated by a sufficiently large power of $\mathbf{m}_\chi = \ker \chi$.

Conventions: For the following, the most interesting cases are those where the components of $\mathbf{a} = (a_1, \dots, a_n)$ belong to the same multiplicative q-orbit; in other words, where there is an $a \in \Bbbk$, such that for each $j = 1, \dots n$, we have $a_j = q^{i_j} a$ for some integer i_j. We will therefore stick to these cases. Moreover, our constructions in fact turn out to be independent of a, so without loss of generality, we chose $a = 1$, i.e. $\mathbf{a} = (q^{i_1}, \dots, q^{i_n})$ with $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$, and write $\chi_{\mathbf{i}}$ for the central character $\chi_{\mathbf{a}}$. If q is an eth root of unity we usually choose the exponents i_j from the representatives $0, \dots, e-1$ for $\mathbb{Z}/e\mathbb{Z}$.

Definition 3.6. From now on for the rest of this paper, we fix $\mathbf{i} \in \mathbb{Z}^n$, viewed as an element of $\mathbb{Z}/e\mathbb{Z}$ if $e \neq 0$.

Definition 3.7. Given a central character $\chi = \chi_{\mathbf{i}}$, we define the *completion* $\widehat{\mathcal{H}}_{\mathbf{i}}$ of $\mathcal{H}$ with respect to powers of the ideal $\mathcal{I}_{\mathbf{m}}$ of $\mathcal{H}$ generated by $\mathbf{m} = \mathbf{m}_\chi$. We have a decomposition

$$\widehat{\mathcal{H}}_{\mathbf{i}} = \bigoplus_{\mathbf{u} \in \mathfrak{S}\mathbf{i}} \widehat{\mathcal{H}}_{\mathbf{i}} e_{\mathbf{u}} \tag{3.8}$$

where

$$\widehat{\mathcal{H}}_{\mathbf{i}} e_{\mathbf{u}} \quad = \quad \left\{ h \in \widehat{\mathcal{H}}_{\mathbf{i}} \,\middle|\, \begin{array}{c} \forall m \in \mathbb{N} \;\exists\; N \in \mathbb{N} \text{ such that} \\ \forall j \in \{1 \dots, n\} h(X_j - q^{u_j})^N \in \ker \mathbf{m}_\chi^m \end{array} \right\} \tag{3.9}$$

The $e_{\mathbf{u}}$ form a complete set of pairwise orthogonal idempotents in $\widehat{\mathcal{H}}_{\mathbf{i}}$.

The following is a direct consequence of (3.3) and (3.7):

Lemma 3.8. *The following sets*

$$\begin{aligned}&\{T_w X_1^{a_1}\cdots X_n^{a_n} e_{\mathbf{u}}\,|\mathbf{u}\in\mathfrak{S}(i_1,\dots,i_n), w\in\mathfrak{S}, a_i\in\mathbb{Z}_{\geq 0}\},\\ &\{T_w X_1^{a_1}\cdots X_n^{a_n} e_{\mathbf{u}}\,|\mathbf{u}\in\mathfrak{S}(i_1,\dots,i_n), w\in\mathfrak{S}, a_i\in\mathbb{Z}_{\leq 0}\},\end{aligned}$$

both form a basis of $\widehat{\mathcal{H}}_{\mathbf{i}}$.

Since $T_r e_{\mathbf{u}} - e_{s_r(\mathbf{u})}T_r \in \bigoplus_{\mathbf{u}\in\mathfrak{S}\mathbf{i}} \Bbbk[[X_1,\dots,X_n]]$, induction on the length of w shows that another topological $\Bbbk$-basis of $\widehat{\mathcal{H}}_{\mathbf{i}}$ is given by

$$\{e_{\mathbf{u}} T_w X_1^{a_1}\cdots X_n^{a_n} \mid \mathbf{u}\in\mathfrak{S}(i_1,\dots,i_n), w\in\mathfrak{S}, a_i\in\mathbb{Z}_{\geq 0}\}$$

and similarly for the version with negative powers of the polynomial generators.

In $\widehat{\mathcal{H}}_{\mathbf{i}}$, we have for for $r = 1,\dots,n-1$ the *intertwining elements*

$$\Phi_r \quad = \quad T_r + \sum_{u_{r+1}\neq u_r} \frac{1-q}{1-X_r X_{r+1}^{-1}} e_{\mathbf{u}} + \sum_{u_{r+1}=u_r} e_{\mathbf{u}}. \tag{3.10}$$

Their properties were, for instance, studied in [BK09].

For each $w\in\mathfrak{S}$ we fix a reduced expression $w = s_{i_1}\cdots s_{i_r}$ and define $\Phi_{[w]} = \Phi_{i_1}\cdots\Phi_{i_r}$. We indicate by $[w]$ that this does depend on the choice of reduced expression. It follows then directly that another topological $\Bbbk$-basis of $\widehat{\mathcal{H}}_{\mathbf{i}}$ is given by

$$\{e_{\mathbf{u}} \Phi_w X_1^{a_1}\cdots X_n^{a_n} \mid \mathbf{u}\in\mathfrak{S}(i_1,\dots,i_n), w\in\mathfrak{S}, a_i\in\mathbb{Z}_{\geq 0}\} \tag{3.11}$$

and similarly for the version with negative powers of the polynomial generators.

Then, similarly to Corollary 3.4, we obtain several topological bases of $\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}}$:

Lemma 3.9. *Any of the following sets is a topological $\Bbbk$-basis of $\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}}$:*

$$\begin{aligned}&\{\mathbf{v}_J e_{\mathbf{u}} T_w X_1^{a_1}\cdots X_n^{a_n} \mid \mathbf{u}\in\mathfrak{S}(i_1,\dots,i_n), w\in D_J, a_i\in\mathbb{Z}_{\geq 0}\},\\ &\{\mathbf{v}_J e_{\mathbf{u}} T_w X_1^{a_1}\cdots X_n^{a_n} \mid \mathbf{u}\in\mathfrak{S}(i_1,\dots,i_n), w\in D_J, a_i\in\mathbb{Z}_{\leq 0}\},\\ &\{\mathbf{v}_J e_{\mathbf{u}} \Phi_w X_1^{a_1}\cdots X_n^{a_n} \mid \mathbf{u}\in\mathfrak{S}(i_1,\dots,i_n), w\in D_J, a_i\in\mathbb{Z}_{\geq 0}\},\\ &\{\mathbf{v}_J e_{\mathbf{u}} \Phi_w X_1^{a_1}\cdots X_n^{a_n} \mid \mathbf{u}\in\mathfrak{S}(i_1,\dots,i_n), w\in D_J, a_i\in\mathbb{Z}_{\leq 0}\}.\end{aligned}$$

Analogously, we obtain bases for $\overline{\mathbf{v}}_J\widehat{\mathcal{H}}_{\mathbf{i}}$ if we replace $\mathbf{v}$ by $\overline{\mathbf{v}}$.

3.4. **A faithful representation of the Hecke algebra.** We now construct a faithful representation of $\mathcal{H}$ respectively $\widehat{\mathcal{H}}_{\mathbf{i}}$ which allows us to realise either algebra as a subalgebra of the endomorphisms of some Laurent polynomial respectively power series ring.

Fix the left ideal $\overline{U} = \left(\sum_{1\leq i\leq n-1}\mathcal{H}(T_i+1)\right)$. We obtain the following slightly sign-adapted version of a result in [Web13]:

Proposition 3.10 (Faithful representation of Hecke algebra I)**.**

i.) *The natural action of $\mathcal{H}$ on $\mathcal{H}/\overline{U}$ by left multiplication is faithful.*
ii.) *This representation is canonically isomorphic to $\Bbbk[X_1^{\pm 1},\dots,X_n^{\pm 1}]\overline{\mathbf{v}}$, where the generators $X_i^{\pm}$, $1\leq i\leq n$, and T_i+1, $1\leq i\leq n-1$ act just by left multiplication respectively by*

$$(T_i+1)f\overline{\mathbf{v}} \quad = \quad \frac{qX_{i+1}-X_i}{X_{i+1}-X_i}(f-s_i(f))\overline{\mathbf{v}}, \tag{3.12}$$

where $s_i(f)$ is the Laurent polynomial f with variables X_{i+1} and X_i interchanged.

Proof. It is a straightforward calculation to verify that the assignments in (ii) define a well-defined action, i.e. that relations (H-1)-(H-7) hold. Moreover, it is clear that the basis (3.3) is mapped to linearly independent endomorphisms. By (3.1) the canonial $\mathcal{H}$-equivariant map $\mu:\mathcal{H}\to\Bbbk[X_1^{\pm 1},\dots,X_n^{\pm 1}]\overline{\mathbf{v}}$, $1\mapsto\overline{\mathbf{v}}$ factors through $\overline{U}$. By (3.3), $\mathcal{H}/\overline{U}=\mathcal{P}$ and μ maps a basis to a basis. Hence it is an isomorphism. Hence it remains to check the formula (3.12). If f is s_i-invariant, then both sides vanish by definition and (3.7). Assume $f=X_ig$ for some s_i-invariant $g\in\Bbbk[X_1^{\pm 1},\dots,X_n^{\pm 1}]$ and abbreviate $h=\frac{qX_{i+1}-X_i}{X_{i+1}-X_i}$. Then we have $(T_i+1)f\overline{\mathbf{v}}=T_iX_iT_iT_i^{-1}g\overline{\mathbf{v}}+X_ig\overline{\mathbf{v}}=qX_{i+1}T_i^{-1}g\overline{\mathbf{v}}+X_ig\overline{\mathbf{v}}=X_{i+1}(T_i+1)g\overline{\mathbf{v}}-(qX_{i+1}-X_i)g\overline{\mathbf{v}}$. Now, by induction, the latter is equal to

$$\begin{aligned} & X_{i+1}h(f-s_i(f))\overline{\mathbf{v}}+\frac{qX_{i+1}X_i-qX_{i+1}^2+X_iX_{i+1}-X_i^2}{X_{i+1}-X_i}g\overline{\mathbf{v}} \\ = \quad & h(X_if-X_{i+1}s_i(f)) \end{aligned}$$

and the claim follows. □

By twisting with the automorphism $\sharp$ from (3.6) we obtain:

Proposition 3.11 (Faithful representation of Hecke algebra II).
Let $U=\sum_{1\leq i<n}\mathcal{H}(T_i-q)$. Then there is a faithful representation of $\mathcal{H}$ on

$$\mathcal{H}/U\cong\Bbbk[X_1^{\pm 1},\dots,X_n^{\pm 1}]\mathbf{v}.$$

Explicitly, the action is given by

$$(T_i-q)f\mathbf{v} \quad = \quad -\frac{qX_i-X_{i+1}}{X_i-X_{i+1}}(f-s_i(f))\mathbf{v}. \tag{3.13}$$

Proof. This follows directly from the fact that $\mathcal{H}/\overline{U}\cong{}^{\sharp}(\mathcal{H}/U)$, the representation $\mathcal{H}/U$ twisted by the automorphism $\sharp$. □

We denote by $\overline{\mathbb{P}(\mathcal{H})}$ respectively $\mathbb{P}(\mathcal{H})$ the faithful representations from (3.12) and (3.13) respectively. The following is immediate.

Corollary 3.12. *We have an isomorphism of $\mathcal{H}$-modules*

$$\overline{\mathbb{P}(\mathcal{H})}\to{}^{\sharp}(\mathbb{P}(\mathcal{H}))\quad\text{given by}\quad f\mathbf{v}\mapsto f^{\sharp}\overline{\mathbf{v}}.$$

Completion gives us faithful representations of the completed algebra:

Corollary 3.13. *i.) There is a faithful representation of $\widehat{\mathcal{H}}_{\mathbf{i}}$ on*

$$\widehat{\overline{\mathbb{P}}(\mathcal{H})}_{\mathbf{i}} = \widehat{\mathcal{H}}_{\mathbf{i}} \otimes_{\mathcal{H}} \mathcal{H}/\overline{U} \cong \bigoplus_{\mathbf{u}\in\mathfrak{S}\mathbf{i}} \Bbbk[[X_1,\dots,X_n]]e_{\mathbf{u}}\overline{\mathbf{v}}$$

by completing the representation from Proposition 3.10.

ii.) There is a faithful representation of $\widehat{\mathcal{H}}_{\mathbf{i}}$ on

$$\widehat{\mathbb{P}(\mathcal{H})}_{\mathbf{i}} = \widehat{\mathcal{H}}_{\mathbf{i}} \otimes_{\mathcal{H}} \mathcal{H}/U \cong \bigoplus_{\mathbf{u}\in\mathfrak{S}\mathbf{i}} \Bbbk[[X_1^{-1},\dots,X_n^{-1}]]e_{\mathbf{u}}\mathbf{v}$$

by completing the representation from Proposition 3.11.

The definitions directly imply the following connection:

Corollary 3.14. *There is an isomorphism of $\widehat{\mathcal{H}}_{\mathbf{i}}$-modules*

$$\widehat{\mathbb{P}(\mathcal{H})}_{\mathbf{i}} \;\cong\; {}^{\sharp}(\widehat{\overline{\mathbb{P}}(\mathcal{H})}_{-\mathbf{i}}) \quad \textit{via} \quad f\mathbf{v} \;\mapsto\; f^{\sharp}\overline{\mathbf{v}},$$

identifying $\Bbbk[[X_1^{-1},\dots,X_n^{-1}]]e_{\mathbf{u}}\mathbf{v}$ with $\Bbbk[[X_1,\dots,X_n]]e_{-\mathbf{u}}\overline{\mathbf{v}}$, where the minus signs applies to all entries, i.e. $-\mathbf{i} = (-i_1,\dots,-i_n)$ and $-\mathbf{u} = (-u_1,\dots,-u_n)$.

Proof. The first statement is clear. The identification follows directly from the fact that $X_i^{\sharp} = X_i^{-1}$ for all i, Corollary 3.13 and the definition in (3.9). $\square$

We finish this section with a few important explicit formulae for the action of the intertwining elements from (3.10).

Lemma 3.15. *For any idempotent $e_{\mathbf{u}}$ as in Definition 3.7, the following equalities hold. For $1 \leq r \leq n-1$, we have*

$$e_{s_r\cdot\mathbf{u}}\Phi_r\overline{\mathbf{v}} \;=\; \begin{cases} \frac{X_r - qX_{r+1}}{X_{r+1}-X_r} e_{s_r\cdot\mathbf{u}}\overline{\mathbf{v}} & \textit{if } u_{r+1}\neq u_r,\\ 0 & \textit{otherwise}, \end{cases} \tag{3.14}$$

and

$$e_{s_r\cdot\mathbf{u}}\Phi_r(X_{r+1}-X_r)\overline{\mathbf{v}} \;=\; \begin{cases} (qX_{r+1}-X_r)e_{s_r\cdot\mathbf{u}}\overline{\mathbf{v}} & \textit{if } u_{r+1}\neq u_r,\\ 2(qX_{r+1}-X_r)e_{\mathbf{u}}\overline{\mathbf{v}} & \textit{if } u_{r+1}= u_r. \end{cases} \tag{3.15}$$

Proof. Since $T_r\overline{\mathbf{v}} = -\overline{\mathbf{v}}$, using the definition (3.10) of Φ_r, we obtain that $e_{s_r\cdot\mathbf{u}}\Phi_r\overline{\mathbf{v}}$ equals $-e_{s_r\cdot\mathbf{u}}\overline{\mathbf{v}} + \frac{1-q}{1-X_rX_{r+1}}e_{s_r\cdot\mathbf{u}}\overline{\mathbf{v}} = \frac{X_r-qX_{r+1}}{X_{r+1}-X_r}e_{s_r\cdot\mathbf{u}}\overline{\mathbf{v}}$ if $u_{r+1}\neq u_r$ and equals zero otherwise. This shows (3.14). On the other hand

$$\begin{aligned} T_r(X_{r+1}-X_r) &= (X_{r+1}+X_r) - 2T_rX_r \\ &= (X_{r+1}-X_r)T_r - 2qX_{r+1}(q^{-1}T_r + q^{-1} - 1). \end{aligned}$$

Hence $T_r(X_{r+1}-X_r)\overline{\mathbf{v}} = (2q-1)X_{r+1} - X_r$. Therefore by (3.10) we obtain

$$e_{s_r\cdot\mathbf{u}}\Phi_r(X_{r+1}-X_r)\overline{\mathbf{v}} \;=\; (2q-1)X_{r+1} - X_r + \begin{cases} (1-q)X_{r+1} & \text{if } u_r\neq u_{r+1},\\ X_{r+1}-X_r & \text{if } u_r = u_{r+1}. \end{cases}$$

Hence the lemma follows. $\square$

Lemma 3.16. *For any idempotent $e_{\mathbf{u}}$ as in Definition 3.7, the following equalities hold: For $1 \leq r \leq n-1$, we have*

$$e_{s_r\cdot\mathbf{u}}\Phi_r\mathbf{v} = \begin{cases} \frac{X_{r+1}-qX_r}{X_{r+1}-X_r} e_{s_r\cdot\mathbf{u}}\mathbf{v} & \text{if } u_{r+1} \neq u_r, \\ (q+1)e_{s_r\cdot\mathbf{u}}\mathbf{v} & \text{if } u_{r+1} = u_r. \end{cases} \tag{3.16}$$

and

$$e_{s_r\cdot\mathbf{u}}\Phi(X_{r+1}-X_r)\mathbf{v} = \begin{cases} (qX_r - X_{r+1})e_{s_r\cdot\mathbf{u}}\mathbf{v} & \text{if } u_{r+1} \neq u_r, \\ (q-1)(X_{r+1}+X_r)e_{\mathbf{u}}\mathbf{v} & \text{if } u_{r+1} = u_r. \end{cases} \tag{3.17}$$

Proof. This is proved analogously to Lemma 3.15. □

4. Affine Schur algebra

In this section we recall the *(affine) Schur algebra* in the version as introduced in [Vig03] and construct a faithful representation for this algebra as well. We describe in detail the basis used by Vignéras. These two tools allow us to give an alternative basis together with a set of algebra generators more in the spirit of the geometric basis of the quiver Schur algebra from [SW11]. This will then finally allow us to connect the two algebras in the last section.

Definition 4.1. The (affine) Schur algebra $\mathcal{S}$ is defined as

$$\mathcal{S} \quad = \quad \mathrm{End}_{\Bbbk[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]}\left(\bigoplus_{J\subseteq\mathbb{I}} \Bbbk[\mathrm{P}^J\backslash\mathrm{G}/\mathrm{I}]\right),$$

where P^J denotes the standard parahoric subalgebra (containing I) attached to J, or alternatively, using the isomorphism from Lemma 3.2 as

$$\mathcal{S} \quad = \quad \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{J\subseteq\mathbb{I}} \mathbf{v}_J\mathcal{H}\right). \tag{4.1}$$

The product of two elements, f, f' in $\mathcal{S}$ is denoted by $f \circ f'$ or just ff'.

We start our study of this algebra by recalling a basis of $\mathcal{S}$ from [Vig03, 4.2.13]:

Lemma 4.2. *A basis of $\mathcal{S}$ is given by*

$$\{\mathbf{b}^d_{K,J} \mid J, K \subseteq \mathbb{I}, d \in D_{K,J}\}, \tag{4.2}$$

where $\mathbf{b}^d_{K,J} \in \mathrm{Hom}_{\mathcal{H}_{\mathbf{i}}}(\mathbf{v}_J\mathcal{H}, \mathbf{v}_K\mathcal{H})$ is defined by

$$\mathbf{b}^d_{K,J}(\mathbf{v}_J) = \sum_{w\in W_K d W_J} T_w. \tag{4.3}$$

Remark 4.3. Note that (4.3) is indeed well-defined, since $\sum_{w\in W_K dW_J} T_w \in \mathbf{v}_K\mathcal{H}$ by Lemma 3.3. Moreover, any element in $\mathrm{Hom}_{\mathcal{H}}(\mathbf{v}_J\mathcal{H}, \mathbf{v}_K\mathcal{H})$, and in particular $\mathbf{b}^d_{K,J}$, is already uniquely determined by its image of $\mathbf{v}_J$.

Example 4.4. If, for instance, $\mathbb{I} = \{s_1, s_2\}$ and $K = \{s_1\}$, $J = \{s_2\}$, then $D_{K,J} \supset D^{\mathbb{I}}_{K,J} = \{1, s_2s_1\}$ and for these two shortest double coset representatives and we have

$$\mathbf{b}^1_{K,J} = 1 + T_1 + T_2 + T_2T_1 \quad \text{and} \quad \mathbf{b}^{s_2s_1}_{K,J} = T_2T_1 + T_1T_2T_1$$

Note that we just sum over all basis elements from a fixed double coset.

Example 4.5. The special morphisms $\mathbf{b}^1_{K,J}$ from (4.2) are easy to describe. Assume $J \subseteq K$. Then we have $\mathbf{b}^1_{K,J}\mathbf{v}_J = \sum_{w\in \mathrm{W}_K} T_w = \mathbf{v}_K$ and $\mathbf{b}^1_{J,K}\mathbf{v}_K = \sum_{w\in \mathrm{W}_K} T_w = \mathbf{v}_J(\sum_{d'\in D^K_{J,\emptyset}} T_{d'})$. Hence $\mathbf{b}^1_{K,J}$ is just the projection sending $\mathbf{v}_J h$ to $\mathbf{v}_K h$ and $\mathbf{b}^1_{J,K}$ is the inclusion sending $\mathbf{v}_K h$ to $\mathbf{v}_J(\sum_{d'\in D^K_{J,\emptyset}} T_{d'})h$, for $h \in \mathcal{H}$.

We like to point out that the labelling of the basis vector $\mathbf{b}^d_{K,J}$ involves a choice d of a shortest double coset representative, although the basis element itself only depends on the coset containing d. In particular the basis can be relabelled when chosing different representatives. If, for instance, for $K, J \subset \mathbb{I}, w \in \mathfrak{S}, p \in \mathfrak{X}$ we define the element $\mathbf{b}^{wp}_{K,J} \in \mathrm{Hom}_{\mathcal{H}_\mathbf{i}}(\mathbf{v}_J\mathcal{H}, \mathbf{v}_K\mathcal{H})$ via

$$\mathbf{b}^{wp}_{K,J}(\mathbf{v}_J) = \sum_{v\in W_K wpW_J} T_v,$$

then $\mathbf{b}^{wp}_{K,J} = \mathbf{b}^d_{K,J}$ for $d \in D_{K,J} \cap W_K wpW_J$, and with the choice of double coset representatives from Section 2.3 we directly obtain the following.

Lemma 4.6. *Both sets*

$$\{\mathbf{b}^{wp}_{K,J} \mid J, K \subseteq \mathbb{I}, w \in D^{\mathbb{I}}_{K,J}, p \in \overrightarrow{\mathfrak{X}}_{d^{-1}K\cap J}\}, \tag{4.4}$$

and

$$\{\mathbf{b}^{wp}_{K,J} \mid J, K \subseteq \mathbb{I}, w \in D^{\mathbb{I}}_{K,J}, p \in \overleftarrow{\mathfrak{X}}_{d^{-1}K\cap J}\} \tag{4.5}$$

form the same basis of $\mathcal{S}$ as in (4.2)*, just labelled differently.*

4.1. **A faithful representation of $\mathcal{S}$.** To construct a faithful representation of the Schur algebra we enlarge the space $\mathbb{P}(\mathcal{H})$.

For any parabolic subgroup W_K of W with $K \subseteq \mathbb{I}$, let $\Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{W_K}$ denote the W_K-invariants under the usual permutation action. We set

$$\mathbb{P}(\mathcal{S})^K \quad = \quad \Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{W_K}\mathbf{v}^{(K)},$$

where $\mathbf{v}^{(K)} = \mathbf{v}$ and the superscript (K) is just a formal index. We have the following characterisation of invariants:

Lemma 4.7. *Let $f\mathbf{v} \in \Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}]\mathbf{v}$ and $K \subseteq \mathbb{I}$. Then*

$$f \in \mathbb{P}(\mathcal{S})^K \quad \Leftrightarrow \quad (T_i - q)f\mathbf{v} = 0 \text{ for all } s_i \in K.$$

Proof. A direct computation shows that, for $s_i \in K$,

$$\begin{aligned}(T_i - q) f\mathbf{v} &= T_i f\mathbf{v} - qf\mathbf{v} \\ &= \left(s_i(f)T_i + (q-1)X_{i+1}\frac{f - s_i(f)}{X_{i+1} - X_i} - qf \right)\mathbf{v} \\ &= \left(q(s_i(f) - f) + (q-1)X_{i+1}\frac{f - s_i(f)}{X_{i+1} - X_i} \right)\mathbf{v} \\ &= \frac{(X_{i+1} - qX_i)(s_i(f) - f)}{X_{i+1} - X_i}\mathbf{v}.\end{aligned}$$

Hence $(T_i - q)f\mathbf{v} = 0$ if and only if $f = s_i(f)$. □

The following is the main theorem of this section.

Theorem 4.8. *There is a faithful representation ρ of $\mathcal{S}$ on*

$$\mathbb{P}(\mathcal{S}) = \bigoplus_{K \subseteq \mathbb{I}} \mathbb{P}(\mathcal{S})^K.$$

In this representation a basis element $\mathbf{b}^d_{K,J}$ of $\mathcal{S}$ acts via

$$\rho(\mathbf{b}^d_{K,J}) f\mathbf{v}^{(J)} \quad = \quad \sum_{a \in D^K_{\emptyset, K\cap dJ}} T_a T_d f \mathbf{v}^{(K)}. \tag{4.6}$$

The proof will follow directly from the next three lemmas. The first of these makes sure that the right hand-side of (4.6) is at least in the correct space.

Lemma 4.9. *For $J, K \subseteq \mathbb{I}, d \in D_{K,J}$ and $f \in \Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathrm{W}_J}$, we have*

$$\sum_{a \in D^K_{\emptyset, K\cap dJ}} T_a T_d f \mathbf{v}^{(K)} \quad \in \quad \mathbb{P}(\mathcal{S})^K. \tag{4.7}$$

Proof. In view of Lemma 4.7, it suffices to check that for all $s_i \in K$

$$\sum_{a \in D^K_{\emptyset, K\cap dJ}} (T_i - q) T_a T_d f \mathbf{v}^{(K)} \quad = \quad 0. \tag{4.8}$$

The left hand side equals

$$\sum_{\substack{a \in D^K_{\emptyset, K\cap dJ} \\ s_i a \in D^K_{\emptyset, K\cap dJ}}} (T_i - q) T_a T_d f \mathbf{v}^{(K)} \quad + \quad \sum_{\substack{a \in D^K_{\emptyset, K\cap dJ} \\ s_i a \notin D^K_{\emptyset, K\cap dJ}}} (T_i - q) T_a T_d f \mathbf{v}^{(K)}. \tag{4.9}$$

Denote by S_1 and S_2 the two summands in (4.9) respectively.

In the first summand S_1, the summands appear in pairs $a, s_i a$. Since we have $(T_i - q)(T_a + T_{s_i a}) = 0$, they cancel and so $S_1 = 0$.

In the second summand S_2, we have $a \in D^K_{\emptyset, K\cap dJ}$ but $s_i a \notin D^K_{\emptyset, K\cap dJ}$. Then Deodhar's Lemma, see e.g. [GP00, Lemma 2.1.2], shows that there exists $s_j \in K \cap dJ$ such that $s_i a = a s_j$, and that, in particular, $l(s_i a) > l(a)$. In

this case, $(T_i - q)T_a = T_a(T_j - q)$. Again using Deodhar's Lemma, we see that $T_jT_d = T_dT_k$ for some $s_k \in d^{-1}K \cap J$, and thus

$$S_2 \in \sum_{s_j \in K \cap dJ} \mathcal{H}(T_j - q)T_d f\mathbf{v}^{(K)} \subseteq \mathcal{H}T_d \sum_{s_k \in J}(T_k - q)f\mathbf{v}^{(K)} = 0$$

by Lemma 4.7, since $f \in \Bbbk[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathrm{W}_J}$. Hence we have proved (4.8). $\square$

Lemma 4.10. *The assignment* (4.6) *defines a representation of* $\mathcal{S}$ *on* $\mathbb{P}(\mathcal{S})$*.*

Proof. It suffices to check that, for basis vectors as in (4.2),

$$\rho(\mathbf{b}^{d_2}_{L,K})\rho(\mathbf{b}^{d_1}_{K,J}) = \rho(\mathbf{b}^{d_2}_{L,K}\mathbf{b}^{d_1}_{K,J}). \tag{4.10}$$

We start with some preparation. Using the basis (4.2) of $\mathcal{S}$, we can write $\mathbf{b}^{d_2}_{L,K}\mathbf{b}^{d_1}_{K,J} = \sum_{d \in D_{L,J}} c_d \mathbf{b}^{d}_{L,J}$ for some coefficients $c_d \in \Bbbk$. Then, on the one hand, we have

$$\mathbf{b}^{d_2}_{L,K}\mathbf{b}^{d_1}_{K,J}(\mathbf{v}_J) = \left(\sum_{d \in D_{L,J}} c_d \sum_{b'' \in D^J_{\emptyset, L \cap dJ}} T_{b''}T_d\right)\mathbf{v}_J.$$

On the other hand, repeatedly using (2.3), we obtain

$$\begin{aligned}
\mathbf{b}^{d_2}_{L,K}\mathbf{b}^{d_1}_{K,J}\mathbf{v}_J &\overset{(4.3)}{=} \mathbf{b}^{d_2}_{L,K} \sum_{w \in W_K d_1 W_J} T_w = \mathbf{b}^{d_2}_{L,K}\mathbf{v}_K T_{d_1} \sum_{a \in D^J_{d_1^{-1}K \cap J, \emptyset}} T_a \\
&\overset{(4.3)}{=} \sum_{w \in W_L d_2 W_K} T_w T_{d_1} \sum_{a \in D^J_{d_1^{-1}K \cap J, \emptyset}} T_a \\
&= \sum_{b \in D^L_{\emptyset, d_2K \cap L}} T_b T_{d_2}\mathbf{v}_K T_{d_1} \sum_{a \in D^J_{d_1^{-1}K \cap J, \emptyset}} T_a \\
&= \left(\sum_{b \in D^L_{\emptyset, d_2K \cap L}} T_b T_{d_2} \sum_{b' \in D^J_{\emptyset, K \cap d_1 J}} T_{b'}T_{d_1}\right)\mathbf{v}_J.
\end{aligned}$$

Therefore,

$$\sum_{b \in D^L_{\emptyset, d_2K \cap L}} T_b T_{d_2} \sum_{b' \in D^J_{\emptyset, K \cap d_1 J}} T_{b'}T_{d_1} - \sum_{d \in D_{L,J}} c_d \sum_{b'' \in D^J_{\emptyset, L \cap dJ}} T_{b''}T_d$$

is contained in $\sum_{s_i \in J} \mathcal{H}(T_i - q)$. To verify formula (4.10), we now calculate

$$\begin{aligned}
\rho(\mathbf{b}^{d_2}_{L,K})\rho(\mathbf{b}^{d_1}_{K,J})f\mathbf{v}^{(J)} &= \rho(\mathbf{b}^{d_2}_{L,K}) \sum_{a \in D^K_{\emptyset, K \cap d_1 J}} T_a T_{d_1} f\mathbf{v}^{(K)} \\
&= \sum_{b \in D^L_{\emptyset, d_2K \cap L}} T_b T_{d_2} \sum_{b' \in D^K_{\emptyset, K \cap d_1 J}} T_{b'}T_{d_1} f\mathbf{v}^{(L)},
\end{aligned} \tag{4.11}$$

where $f \in \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathrm{W}_J}$, and

$$\sum_{d \in D_{L,J}} c_d \rho(\mathbf{b}^d_{L,J}) f \mathbf{v}^{(J)} = \sum_{d \in D_{L,J}} c_d \sum_{b'' \in D^J_{\emptyset, L \cap dJ}} T_{b''} T_d f \mathbf{v}^{(L)}. \tag{4.12}$$

Taking the difference of (4.11) and (4.12) we obtain

$$\begin{aligned} &\big(\rho(\mathbf{b}^{d_2}_{L,K})\rho(\mathbf{b}^{d_1}_{K,J}) - \sum_{d \in D_{L,J}} c_d \rho(\mathbf{b}^d_{L,J})\big) f \mathbf{v}^{(J)} \\ = &\left(\sum_{b \in D^L_{\emptyset, d_2 K \cap L}} T_b T_{d_2} \sum_{b' \in D^J_{\emptyset, K \cap d_1 J}} T_{b'} T_{d_1} - \sum_{d \in D_{L,J}} c_d \sum_{b'' \in D^J_{\emptyset, L \cap dJ}} T_{b''} T_d \right) f \mathbf{v}^{(L)}. \end{aligned}$$

By the above, this is, however, contained in $\sum_{s_i \in J} \mathcal{H}(T_i - q) f \mathbf{v}^{(L)}$ and hence must be zero by Lemma 4.7, as $f \in \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathrm{W}_J}$. $\square$

Lemma 4.11. *The representation* ρ *from* (4.6) *is faithful.*

Proof. Let $J, K \subset \mathbb{I}$ and $Z = \sum_{\substack{J,K \subseteq \mathbb{I} \\ d \in D_{K,J}}} c_d \, \rho(\mathbf{b}^d_{K,J})$ with arbitrary $c_d \in \Bbbk$. Then it is enough to show that $Z = 0$ implies $\sum_{d \in D_{K,J}} c_d \mathbf{b}^d_{K,J} = 0$ for each pair $J, K \subset \mathbb{I}$. We start with the following observation. Given $h \in \mathcal{H}$ and $L \subset \mathbb{I}$ then

$$h \, \mathbb{P}(\mathcal{S})^L = 0 \quad \text{implies} \quad h \left(\sum_{w \in \mathrm{W}_L} T_w \right) = 0. \tag{4.13}$$

Indeed, assume $h \, \mathbb{P}(\mathcal{S})^L = 0$. Since $(T_i - q) \sum_{w \in \mathrm{W}_L} T_w = 0$ for any $s_i \in L$, Lemma 4.7 yields that

$$\sum_{w \in \mathrm{W}_L} T_w \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \mathbf{v} \subseteq \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathrm{W}_L} \mathbf{v}.$$

In particular,

$$\left(h \sum_{w \in \mathrm{W}_L} T_w \right) \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \mathbf{v} \quad \subseteq h \, \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{\mathrm{W}_L} \mathbf{v} \overset{\text{(ass.)}}{=} 0.$$

Together with the faithfulness of the representation of $\mathcal{H}$ on $\Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \mathbf{v}$ in Lemma 3.11, the claim in(4.13) follows.

Now suppose that $Z = 0$. Projecting onto the different summands of $\mathbb{P}(\mathcal{S})$ gives that for any $J, K \subseteq \mathbb{I}$ we have

$$\sum_{d \in D_{K,J}} c_d \rho(\mathbf{b}^d_{K,J}) \mathbb{P}(\mathcal{S})^J \quad = \quad \left(\sum_{d \in D_{K,J}} c_d \sum_{a \in D^J_{\emptyset, K \cap dJ}} T_a T_d \right) \mathbb{P}(\mathcal{S})^K = 0.$$

Observation (4.13) implies that

$$\begin{aligned} 0 &= \sum_{d\in D_{K,J}} c_d \sum_{a\in D^J_{\emptyset,K\cap dJ}} T_aT_d \sum_{w\in \mathrm{W}_J} T_w \overset{(3.5)}{=} \sum_{d\in D_{K,J}} c_d \sum_{a\in D^J_{\emptyset,K\cap dJ}} T_aT_d\mathbf{v}_J \\ &= \sum_{d\in D_{K,J}} c_d\mathbf{b}^d_{K,J}\mathbf{v}_J. \end{aligned}$$

We conclude $\sum_{d\in D_{K,J}} c_d\mathbf{b}^d_{K,J} = 0$ for every $J, K \subseteq \mathbb{I}$, which completes the proof. □

Theorem 4.8 is proved.

4.2. **Generators and (some) relations for $\mathcal{S}$.** In this section, we determine a nice generating set for the algebra $\mathcal{S}$. We start with a few technical tools.

Lemma 4.12. *Let $K_1, K_2 \subseteq \mathbb{I}, d \in D_{K_2,K_1}$ and let $J = K_1 \cap d^{-1}K_2$. Then*

$$\mathbf{b}^d_{K_2,K_1} = \mathbf{b}^1_{K_2,dJ}\mathbf{b}^d_{dJ,J}\mathbf{b}^1_{J,K_1}. \tag{4.14}$$

Proof. We first note that for $|J| = |dJ|$ (which holds for J as in the lemma),

$$\mathbf{b}^d_{dJ,J}\mathbf{v}_J = \mathbf{v}_{dJ}T_d \sum_{b\in D^J_{J\cap J,\emptyset}} T_b = \mathbf{v}_{dJ}T_d \tag{4.15}$$

and that, if $J \subseteq K$, then

$$\mathbf{b}^1_{K,dJ}\mathbf{v}_{dJ} = \sum_{w\in \mathrm{W}_K 1 \mathrm{W}_{dJ}} T_w = \mathbf{v}_K. \tag{4.16}$$

We now apply the right hand side of (4.14) to $\mathbf{v}_{K_1}$ and deduce

$$\begin{aligned} \mathbf{b}^1_{K_2,dJ}\mathbf{b}^d_{dJ,J}\mathbf{b}^1_{J,K_1}(\mathbf{v}_{K_1}) &= \mathbf{b}^1_{K_2,dJ}\mathbf{b}^d_{dJ,J}\left(\sum_{w\in \mathrm{W}_{K_1}} T_w\right) \\ = \mathbf{b}^1_{K_2,dJ}\mathbf{b}^d_{dJ,J}\left(\mathbf{v}_J \sum_{a\in D^{K_1}_{J,\emptyset}} T_a\right) &= \mathbf{b}^1_{K_2,dJ}\left(\mathbf{v}_{dJ}T_d \sum_{a\in D^{K_1}_{J,\emptyset}} T_a\right) \\ \overset{(4.16)}{=} \mathbf{v}_{K_2}T_d \sum_{a\in D^{K_1}_{d^{-1}K_2\cap K_1,\emptyset}} T_a &= \mathbf{b}^d_{K_2,K_1}\mathbf{v}_{K_1} \end{aligned}$$

as desired. □

As a consequence we obtain the following.

Corollary 4.13. *The Schur algebra $\mathcal{S}$ is generated (as an algebra) by*

$$\{\mathbf{b}^1_{K,J}, \mathbf{b}^w_{wJ,J} \mid J, K \subseteq \mathbb{I}, w \in \mathrm{W} \textit{ with } |J| = |wJ|\}. \tag{4.17}$$

Based on this, we will give another generating set in Proposition 6.19.

4.3. **The centre of $\mathcal{S}$.** In this section we determine the centre of the Schur algebra. It turns out that the centre is just given by multiplication with elements from the centre (3.7) of the Hecke algebra.

Lemma 4.14. *The centre $Z(\mathcal{S})$ of $\mathcal{S}$ equals $Z(\mathcal{H})$ in the sense that*

$$Z(\mathcal{S}) = \{\cdot z \mid z \in Z(\mathcal{H})\} \quad \subseteq \quad \mathrm{End}_{\mathcal{H}}(\bigoplus_{J\subseteq\mathbb{I}} \mathbf{v}_J\mathcal{H}) = \mathcal{S}$$

Proof. For the inclusion $\supseteq$, it is clear that multiplication with $z \in Z(\mathcal{H})$ commutes with any $\mathcal{H}$-endomorphism of $\bigoplus_{J\subseteq\mathbb{I}} \mathbf{v}_J\mathcal{H}$ and hence belongs to $\mathcal{S}$. It furthermore commutes with any element in $\mathcal{S}$ and hence belongs to $Z(\mathcal{S})$.

For the inclusion $\subseteq$, let $f \in Z(\mathcal{S})$ and test with the generators given in (4.17). For $\mathbf{b}^d_{dJ,J}$ we see that

$$\mathbf{b}^d_{dJ,J} \circ f\left(\sum_{K\subseteq\mathbb{I}} \mathbf{v}_K\right) = \mathbf{b}^d_{dJ,J}\left(\sum_{L\in\subseteq\mathbb{I}} \mathbf{v}_L h_L\right) = \mathbf{b}^d_{dJ,J}\left(\sum_{L\in\subseteq\mathbb{I}} \mathbf{v}_L\right) h_L \overset{(4.15)}{=} \mathbf{v}_{dJ}T_d h_J$$

for some $h_L \in \mathcal{H}$. On the other hand, since f is central,

$$\mathbf{b}^d_{dJ,J} \circ f\left(\sum_{K\subseteq\mathbb{I}} \mathbf{v}_K\right) = f \circ \mathbf{b}^d_{dJ,J}\left(\sum_{K\subseteq\mathbb{I}} \mathbf{v}_K\right) \overset{(4.15)}{=} f(\mathbf{v}_{dJ}T_d) = f(\mathbf{v}_{dJ})T_d.$$

By comparing the two formulae, $f(\mathbf{v}_J) \in \mathbf{v}_{dJ}\mathcal{H}$ and thus $f \in \sum_{J\subseteq\mathbb{I}} \mathrm{End}_{\mathcal{H}}(\mathbf{v}_J\mathcal{H})$ is a diagonal endomorphism. Therefore, for any $K \subseteq \mathbb{I}$,

$$f(\mathbf{v}_L) = \mathbf{v}_L h_L, \quad \text{and moreover} \quad h_{dJ}T_d = T_d h_J, \tag{4.18}$$

again by comparing the above two formulae.

Now we test with $\mathbf{b}^1_{K,J}$ for $J \subseteq K$ and obtain

$$f \circ \mathbf{b}^1_{K,J}\left(\sum_{L\subseteq\mathbb{I}} \mathbf{v}_L\right) = f \circ \mathbf{b}^1_{K,J}(\mathbf{v}_J) \overset{(4.16)}{=} f(\mathbf{v}_K) \overset{(4.18)}{=} \mathbf{v}_K h_K.$$

Since f is central, this equals

$$\mathbf{b}^1_{K,J} \circ f\left(\sum_{L\subseteq\mathbb{I}} \mathbf{v}_L\right) \overset{(4.18)}{=} \sum_{L\subseteq\mathbb{I}} \mathbf{b}^1_{K,J}(\mathbf{v}_L h_L) = \sum_{L\subseteq\mathbb{I}} \mathbf{b}^1_{K,J}(\mathbf{v}_L) h_L = \mathbf{v}_K h_J.$$

Since all $J \subseteq \mathbb{I}$ contain $\emptyset$, this yields, for all $J, K \subseteq \mathbb{I}$, the equality

$$h_J = h_K =: h.$$

Using (4.18) we obtain $T_d h = hT_d$ for all $T_d \in \mathcal{H}$, and thus $f = \cdot h \in Z(\mathcal{H})$. $\square$

4.4. **Another basis for $\mathcal{S}$.** The main goal of this section is to construct an alternative basis of $\mathcal{S}$ which mimics the geometric basis of the quiver Schur algebra defined in [SW11]. We start by investigating the sets D_{K_2,K_1} further.

Lemma 4.15. *Assume $K \subseteq \mathbb{I}$ and suppose $v = wp$ with $w \in \mathfrak{S}$ and $p = X_1^{a_1} \cdots X_n^{a_n} \in \mathfrak{X}$ with $a_i \leq 0$ for $i = 1, \dots n$. Then $T_v = \sum_{u \in \mathfrak{S}} c_u T_u p$, for some coefficients $c_u \in \mathbb{k}$.*

Proof. Note first that, using induction on the degree of p and commutativity of $X_1, \dots, X_n$, it suffices to check this for $p = X_i^{-1}$. So suppose $v = wX_i^{-1}$. Then by (2.2) $v = ws_{i-1}s_{i-2} \cdots s_1 \tau^{-1} s_{n-1}s_{n-2} \cdots s_i$, and thus v has a reduced expression $\tilde{v}\tau^{-1}s_{n-1} \cdots s_i$ with some $\tilde{v} \in \mathfrak{S}$. Thus $T_v = T_{\tilde{v}} T_\tau^{-1} T_{n-1} \cdots T_i = aT_{\tilde{v}} T_1 T_2 \cdots T_{i-1} X_i^{-1}$ where a is a power of q. Writing $aT_{\tilde{v}} T_1 T_2 \cdots T_{i-1} = \sum_{u \in \mathfrak{S}} c_u T_u$, the claim follows. □

Lemma 4.16. *Let $d \in D_{K_2,K_1}$, $J = K_1 \cap d^{-1}K_2$ and write $d = wp$ with $w \in \mathfrak{S}$ and $p \in \mathfrak{X}$. Then $wJ = dJ$ and $p \in \mathfrak{X}^{\mathrm{W}_J}$.*

Proof. We have $d(K_1 \cap d^{-1}K_2) = dK_1 \cap K_2$ and thus for any $i \in (K_1 \cap d^{-1}K_2)$, $ds_id^{-1} = s_j$ for some $j \in dK_1 \cap K_2$. On the other hand,

$$ds_id^{-1} = wps_ip^{-1}w^{-1} = ws_iw^{-1}ws_ips_ip^{-1}w^{-1} = vh,$$

where $v = ws_iw^{-1} \in \mathfrak{S}$ and $h = ws_ips_ip^{-1}w^{-1} \in \mathfrak{X}$, hence $v = s_j, h = 1$. Now $ws_ips_ip^{-1}w^{-1} = 1$ if and only if $s_ips_ip^{-1} = 1$ if and only if $p \in \mathfrak{X}^{\mathrm{W}_{K_1 \cap d^{-1}K_2}} = \mathfrak{X}^{\mathrm{W}_J}$. Moreover, $v = ws_iw^{-1} = s_j$ implies $wJ = dJ$. □

Let d, w, p be as in Lemma 4.16 and let $d' \in D_{dJ,J}$ be the shortest double coset representative for the coset of w. Since p commutes with W_J, the $(\mathrm{W}_{dJ}, \mathrm{W}_J)$-double coset defined by $d = wp$ is the same as the one defined by $d'p$. Hence, by the previous paragraph, and (4.14), any basis element can be written as

$$\mathbf{b}_{K_2,K_1}^{d} = \mathbf{b}_{K_2,K_1}^{wp} = \mathbf{b}_{K_2,dJ}^{1} \mathbf{b}_{dJ,J}^{d} \mathbf{b}_{J,K_1}^{1} = \mathbf{b}_{K_2,dJ}^{1} \mathbf{b}_{dJ,J}^{d'p} \mathbf{b}_{J,K_1}^{1} \tag{4.19}$$

for $J = d^{-1}K_2 \cap K_1$.

Keeping this notation, we next observe that, using Lemma 4.15,

$$\begin{aligned} \mathbf{b}_{dJ,J}^{d'p}(\mathbf{v}_J) &= \mathbf{v}_{dJ} T_{wp} = \mathbf{v}_{dJ} \left(\sum_{u \in \mathfrak{S}} c_u T_u \right) p \\ &= T_{wp} \mathbf{v}_J = \left(\sum_{u \in \mathfrak{S}} c_u T_u \right) p \mathbf{v}_J = \left(\sum_{u \in \mathfrak{S}} c_u T_u \right) \mathbf{v}_J p \end{aligned} \tag{4.20}$$

where, in the last equality, we have used that $p \in \mathfrak{X}^{\mathrm{W}_J}$ by Lemma 4.16. Hence

$$\mathbf{v}_{dJ} \left(\sum_{u \in \mathfrak{S}} c_u T_u \right) = \left(\sum_{u \in \mathfrak{S}} c_u T_u \right) \mathbf{v}_J$$

meaning that left multiplication by $(\sum_{u \in \mathfrak{S}} c_u T_u)$ is in $\mathrm{Hom}_{\mathcal{H}}(\mathbf{v}_J\mathcal{H}, \mathbf{v}_{dJ}\mathcal{H})$.

Using these relations, we will now give a different basis of $\mathcal{S}$, which will be more convenient to work with later on.

Proposition 4.17. *Let $K_1, K_2 \subset \mathbb{I}$. Then the set*

$$\mathcal{B}_{K_2,K_1} = \left\{ \mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\mathbf{b}^p_{J,J}\mathbf{b}^1_{J,K_1} \;\middle|\; \begin{array}{l} J = K_1 \cap w^{-1}K_2, \\ w \in D^{\mathbb{I}}_{K_2,K_1}, p \in \overrightarrow{\mathfrak{X}_J} \end{array} \right\}, \tag{4.21}$$

is a basis of the space of homomorphisms $\mathrm{Hom}_{\mathcal{H}}(\mathbf{v}_{K_1}\mathcal{H}, \mathbf{v}_{K_2}\mathcal{H})$ and hence

$$\mathcal{B}^{\mathcal{S}} = \bigcup_{(K_1,K_2)} \mathcal{B}_{K_1,K_2} \tag{4.22}$$

is a basis of the Schur algebra $\mathcal{S}$.

Proof. We first compute the evaluation of these elements on $\mathbf{v}_{K_1}$. We have

$$\begin{aligned} \mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\mathbf{b}^p_{J,J}\mathbf{b}^1_{J,K_1}(\mathbf{v}_{K_1}) &= \mathbf{b}^1_{K_2,wJ}(T_w b^p_{J,J}\mathbf{v}_J \sum_{a\in D^{K_1}_{J,\emptyset}} T_a) \\ = \mathbf{b}^1_{K_2,wJ}(T_w \sum_{b\in D^J_{\emptyset,J\cap p'J}} T_b T_{p'}\mathbf{v}_J \sum_{a\in D^{K_1}_{J,\emptyset}} T_a) & \\ = \mathbf{b}^1_{K_2,wJ}\left(\sum_{b\in D^{wJ}_{\emptyset,wJ\cap wp'J}} T_b T_w T_{p'}\mathbf{v}_{K_1} \right) &= \mathbf{b}^1_{K_2,wp'J}(T_w T_{p'}\mathbf{v}_{K_1}), \end{aligned}$$

where p' is the unique element in $\mathrm{W}_J p \mathrm{W}_J \cap D_{J,J}$. Now, let d be the unique element in $\mathrm{W}_{K_2} wp' \mathrm{W}_{K_1} \cap D_{K_2,K_1}$ and write $wp' = dv$ with $v \in \mathrm{W}_{K_1}$ and $l(dv) = l(d) + l(v)$.

We would like to show that

$$\mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\mathbf{b}^p_{J,J}\mathbf{b}^1_{J,K_1} = c_d\mathbf{b}^d_{K_2,K_1} + \sum_{d'\in D_{K_2,K_1}, d'>d} c_{d'}\mathbf{b}^{d'}_{K_2,K_1} \tag{4.23}$$

for some $c_{d'} \in \mathbb{k}$ with $c_d \neq 0$. Then the Proposition follows from Lemma 4.2.

Claim 1: For any $x, y \in \mathrm{W}$ we have $T_x T_y = q^{a(x,y)}T_{xy} + (q-1)\sum_{z>xy} c_z T_z$ for some $c_z \in \mathbb{k}$ and $a(x,y) = \frac{1}{2}(l(x) + l(y) - l(xy))$.

Proof. This formula is deduced in the proof of [Mat99, Proposition 1.16] for lengths instead of Bruhat orders and only for $\mathfrak{S}$. However, replacing the permutation realisation of $\mathfrak{S}$ by the realisation of W as permutations on $\mathbb{Z}$ (see [Gre02]) to define the sets $N(x)$ used in [Mat99, Proposition 1.16], the proof generalises verbatim to our situation. □

Claim 2: Let $u \in \mathrm{W}$ with $u > wp'$. Letting d_u denote the unique element in $\mathrm{W}_{K_2} u \mathrm{W}_{K_1} \cap D_{K_2,K_1}$, we have $d_u \geq d$.

Proof. By definition $u = a_2 d_u a_1$ for some unique $a_i \in K_i$ and we can choose a reduced expression of u compatible with this decomposition. Now $wp' < u$ means wp' can be obtained by deleting some of the simple reflections. In particular, $d \leq d_u$. $\square$

From Claim 1 we see that

$$\begin{aligned}\mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\mathbf{b}^p_{J,J}\mathbf{b}^1_{J,K_1}(\mathbf{v}_{K_1}) \quad &= \quad \mathbf{b}^1_{K_2,wp'J}(T_w T_{p'}\mathbf{v}_{K_1}) \\ = \quad q^a \mathbf{b}^1_{K_2,wp'J}&\left((T_{wp'}\mathbf{v}_{K_1} + (q-1)\textstyle\sum_{z>xy} c_z T_z)\mathbf{v}_{K_1}\right)\end{aligned}$$

for some $c_z \in \Bbbk$ and $a = a(w,p')$. Now Claim 2 implies that, when rewriting this in the basis of the $\mathbf{b}^{d'}_{K_2,K_1}$ only basis elements indexed with $d' \geq d$ occur. Moreover, the leading term $T_{wp'}$ contributes $q^{\frac{1}{2}(l(w)+l(p')-l(wp'))+l(v)}$ to the coefficient of $\mathbf{b}^d_{K_2,K_1}$ while any other T_z that might contribute to the coefficient has coefficient of the form $c(q-1)^a$ for some integer a and nonzero scalar c, so the coefficient of $\mathbf{b}^d_{K_2,K_1}$ is nonzero and (4.23) follows. We conclude that the set given in (4.22) is indeed a basis for $\mathcal{S}$. $\square$

Remark 4.18. In the faithful representation ρ from Theorem 4.8, a basis element $\mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\mathbf{b}^p_{J,J}\mathbf{b}^1_{J,K_1}$ as in Proposition 4.21 acts by sending $f\mathbf{v}^{(K_1)}$ to

$$\left(\sum_{a\in D^{K_2}_{\emptyset,wJ}} T_a\right) T_w g_p \left(\sum_{b\in D^{K_1}_{\emptyset,J}} T_b\right) f\mathbf{v}^{(K_2)}.$$

where g_p is defined in Lemma 4.19 below.

4.5. **The subalgebra $\mathcal{Q}$.** We now construct an important commutative subalgebra $\mathcal{Q}$ of $\mathcal{S}$. For this let $J \subset \mathbb{I}$ and recall the notation from Section 2.3.

Lemma 4.19. *For $p \in \overrightarrow{\mathfrak{X}_J}$ and $J \subseteq \mathbb{I}$ we have*

$$\mathbf{b}^p_{J,J}(\mathbf{v}_J) \quad = \quad g_p \mathbf{v}_J \tag{4.24}$$

where g_p is a scalar multiple of the sum $\sum_{x\in \mathrm{W}_J} x(p)$ over all monomials in the W_J-orbit of p.

Proof. On the one hand, $\mathbf{b}^p_{J,J}(\mathbf{v}_J) = \sum_{u\in \mathrm{W}_J p \mathrm{W}_J} T_u$. Noting that

$$\mathrm{W}_J p \mathrm{W}_J = \{vf \mid v \in \mathrm{W}_J, f = \sigma p \sigma^{-1} \text{for some } \sigma \in \mathrm{W}_J\}$$

we see that $\mathbf{b}^p_{J,J}\mathbf{v}_K = \sum_{v\in \mathrm{W}_K, p'\in \mathrm{W}_J\cdot p} T_{vp'}$, which, using Lemma 4.15 can be expressed as $\sum_{v\in\mathfrak{S}} T_v f_v$ for some polynomials f_v, all of whose monomial terms are conjugate to p under W_J.

On the other hand, writing $p = td = dt'$ for $d \in D_{J,J}, t, t' \in \mathrm{W}_J$, we have

$$\begin{aligned}\mathbf{b}^p_{J,J}(\mathbf{v}_J) \quad &= \mathbf{v}_J T_d(\textstyle\sum_{a\in D^J_{d^{-1}J\cap J}} T_a) &&= \mathbf{v}_J q^{-l(t)} T_t T_d \textstyle\sum_{a\in D^J_{d^{-1}J\cap J}} T_a \\ &= \mathbf{v}_J(q^{-l(t)} T_p \textstyle\sum_{a\in D^J_{d^{-1}J\cap J}} T_a) &&= \mathbf{v}_J q^{-l(t)} p \textstyle\sum_{a\in D^J_{d^{-1}J\cap J}} T_a \\ &= \mathbf{v}_J A \in \mathbf{v}_J \mathcal{H}_J\end{aligned}$$

and similarly

$$\begin{aligned}
\mathbf{b}^p_{J,J}(\mathbf{v}_J) &= \textstyle\sum_{a'\in D^J_{J\cap J}} T_{a'}T_d\mathbf{v}_J && = q^{-l(t')}\textstyle\sum_{a'\in D^J_{dJ\cap J}} T_{a'}T_dT_{t'}\mathbf{v}_J \\
&= q^{-l(t')}\textstyle\sum_{a'\in D^J_{d^{-1}J\cap J}} T_{a'}T_p\mathbf{v}_J && = q^{-l(t')}\textstyle\sum_{a'\in D^J_{dJ\cap J}} T_{a'}p\mathbf{v}_J \\
&= A'\mathbf{v}_J \in \mathbf{v}_J\mathcal{H}_J.
\end{aligned}$$

Since $T_i\mathbf{v}_J = \mathbf{v}_JT_i = q\mathbf{v}_J$ for all $s_i \in J$, moreover $A, A' \in \Bbbk[X_1^{\pm1}, \dots X_n^{\pm1}]$. Observing $\mathbf{v}_JAT_i = \mathbf{b}^p_{J,J}\mathbf{v}_JT_i = A'\mathbf{v}_JT_i = qA'\mathbf{v}_J$ for all $s_i \in J$, we furthermore deduce, using Lemma 4.7, that $A, A' \in \Bbbk[X_1^{\pm1}, \dots X_n^{\pm1}]^{\mathrm{W}_J}$ and hence $A = A'$.

From the two preceding paragraphs, we see that

$$\mathbf{b}^p_{J,J}(\mathbf{v}_J) = \mathbf{v}_JA \in \mathbf{v}_J\Bbbk[X_1^{\pm1}, \dots X_n^{\pm1}]^{\mathrm{W}_J}$$

and all summands in A are W_K-conjugate to p. Therefore $\mathbf{b}^p_{J,J}(\mathbf{v}_J)$ is a scalar multiple of $g_p\mathbf{v}_J$, as claimed. □

Proposition 4.20. *The $\Bbbk$-vector space spanned by the $b^p_{J,J}$, for $J \subset \mathbb{I}$, $p \in \overrightarrow{\mathfrak{X}_J}$, forms a commutative subalgebra $\mathcal{Q}$ of $\mathcal{S}$ which contains the centre $Z(\mathcal{S})$ of $\mathcal{S}$.*

Proof. This follows directly from Lemmas 4.19 and 4.14 via Remark 4.3. □

Remark 4.21. Note that by Lemma 4.19 and Lemma 4.7, the subalgebra $\mathcal{Q}$ consists of precisely those elements in $\mathrm{Hom}_{\mathcal{H}}(\mathbf{v}_J\mathcal{H}, \mathbf{v}_J\mathcal{H})$ (for some $J \subseteq \mathbb{I}$) which are given by left multiplication with some $f \in \mathcal{P} \subset \mathcal{H}$ which satisfies $(T_i - q)f\mathbf{v} = 0$ for all $i \in J$.

Remark 4.22. The elements $b^p_{J,J}$, for $J \subset \mathbb{I}$, $p \in \overrightarrow{\mathfrak{X}_J}$, are in fact linearly independent by Lemma 4.19, hence form a basis of $\mathcal{Q}$. As an algebra, $\mathcal{Q}$ is a direct sum of algebras indexed by $J \subset \mathbb{I}$ with factors isomorphic to $\Bbbk[X_1^{\pm1}, \dots, X_n^{\pm1}]^{\mathrm{W}_J}$. The centre $Z(\mathcal{S}) = \Bbbk[X_1^{\pm1}, \dots, X_n^{\pm1}]^{\mathfrak{S}}$, see Lemma 4.14, embeds diagonally.

Lemma 4.23. *The $\Bbbk$-vector space $\mathcal{S}$ carries the structure of a finitely generated free $\mathcal{Q}$-module on basis*

$$\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}} = \left\{ \mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\mathbf{b}^1_{J,K_1} \;\middle|\; \begin{array}{l} K_1, K_2 \subseteq \mathbb{I}, w \in D^{\mathbb{I}}_{K_2,K_1}, \\ J = K_1 \cap w^{-1}K_2 \end{array} \right\}.$$

(Note that we do not claim here that $\mathcal{S}$ is a free $\mathcal{Q}$-module by restriction of the regular action.)

Proof. We define the action of a basis element $\mathbf{b}^{p'}_{K,K} \in \mathcal{Q}$ on a basis element $\mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\mathbf{b}^p_{J,J}\mathbf{b}^1_{J,K_1} \in \mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}$ (for some $K_1, K_2 \subseteq \mathbb{I}$) by

$$\mathbf{b}^{p'}_{K,K} \circledast \mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\mathbf{b}^p_{J,J}\mathbf{b}^1_{J,K_1} \quad = \quad \begin{cases} \mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\mathbf{b}^{pp'}_{J,J}\mathbf{b}^1_{J,K_1} & \text{if } J = K, \\ 0 & \text{otherwise.} \end{cases}$$

By Remark 4.22 this is a well-defined action of $\mathcal{Q}$. Obviously, the module is generated by $\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}$. Freeness then follows from Remark 4.22 and Proposition 4.17. □

4.6. **The twisted faithful representation of the Schur algebra.** The automorphism $\sharp$ from (3.6) allows us to define the Schur algebra using $\overline{\mathbf{v}}_K\mathcal{H}$ instead of $\mathbf{v}_K\mathcal{H}$ via the obvious isomorphism of algebras

$$\mathcal{S} \cong \mathrm{Hom}_{\mathcal{H}} \bigoplus_{J,K\subseteq \mathbb{I}} ((\mathbf{v}_J\mathcal{H})^\sharp, (\mathbf{v}_K\mathcal{H})^\sharp) = \mathrm{Hom}_{\mathcal{H}} \bigoplus_{J,K\subseteq \mathbb{I}} (\overline{\mathbf{v}}_J\mathcal{H}, \overline{\mathbf{v}}_K\mathcal{H}).$$

Similarly to Section 3.4, we also have a faithful representation $\overline{\rho}$ of $\mathcal{S}$ on

$$\overline{\mathbb{P}}(\mathcal{S}) \quad = \quad \bigoplus_{K\subseteq\mathbb{I}} \Bbbk[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{W_K}\overline{\mathbf{v}}^{(K)}, \tag{4.25}$$

where again the superscript on $\overline{\mathbf{v}}^{(K)}$ is just a book-keeping device, given by

$$\overline{\rho}(\mathbf{b}^d_{K,J})f\overline{\mathbf{v}}^{(J)} = \sum_{a\in D^K_{\emptyset, K\cap dJ}} T_a^\sharp T_d^\sharp f\overline{\mathbf{v}}^{(K)}.$$

We also have the following analogue to Corollary 3.12

Corollary 4.24. *We have an isomorphism of representation*

$$\overline{\mathbb{P}}(\mathcal{S}) \cong {}^\sharp\mathbb{P}(\mathcal{S}) \quad \textit{given by} \quad f\overline{\mathbf{v}}^{(K)} \mapsto f^\sharp\mathbf{v}^{(K)}.$$

5. A completion of $\mathcal{S}$

Recall the character $\chi = \chi_{\mathbf{i}}$ for our fixed $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ from Definition 3.6, and the ideals $\mathbf{m}_\chi$ and $\mathcal{I}_m$ in $Z(\mathcal{H})$ respectively $\mathcal{H}$ from Section 3.3. By Lemma 4.14, we can identify the centre of $\mathcal{S}$ with the centre of $\mathcal{H}$. Define $\widehat{\mathcal{S}}_{\mathbf{i}}$ to be the completion of $\mathcal{S}$ at the nested sequence of ideals

$$\mathcal{J}_m \quad = \quad \bigoplus_{J,K\subseteq\mathbb{I}} \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}^J, \mathbf{v}_K\mathcal{I}_m) \tag{5.1}$$

in $\mathcal{S}$ generated by the maximal ideal $\mathbf{m}_\chi$ of $Z(\mathcal{S})$.

5.1. **Compatibility with the completion of $\mathcal{H}$.** The following gives an alternative definition of $\widehat{\mathcal{S}}_{\mathbf{i}}$, analogous to (4.1), using the completed Hecke algebras.

Proposition 5.1. *There is an isomorphism of algebras*

$$\widehat{\mathcal{S}}_{\mathbf{i}} \cong \mathrm{End}_{\widehat{\mathcal{H}}_{\mathbf{i}}}\left(\bigoplus_{J\subseteq\mathbb{I}} \mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}}\right).$$

Proof. We have

$$\begin{aligned}
&\mathrm{End}_{\widehat{\mathcal{H}}_{\mathbf{i}}}\left(\bigoplus_{J\subseteq \mathbb{I}\}}\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}}\right) \cong \bigoplus_{J,K\subseteq\mathbb{I}}\mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}\left(\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}},\mathbf{v}_K\widehat{\mathcal{H}}_{\mathbf{i}}\right)\\
=& \bigoplus_{J,K\subseteq\mathbb{I}}\mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}\left(\varprojlim \mathbf{v}_J\mathcal{H}/\mathcal{I}_m,\varprojlim \mathbf{v}_K\mathcal{H}/\mathcal{I}_k\right)\\
\cong& \bigoplus_{J,K\subseteq\mathbb{I}}\varprojlim\mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}\left(\varprojlim \mathbf{v}_J\mathcal{H}/\mathcal{I}_m,\mathbf{v}_K\mathcal{H}/\mathcal{I}_k\right)\\
\cong& \bigoplus_{J,K\subseteq\mathbb{I}}\varprojlim\mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}\left(\mathbf{v}_J\mathcal{H}/\mathcal{I}_k,\mathbf{v}_K\mathcal{H}/\mathcal{I}_k\right)\\
\cong& \bigoplus_{J,K\subseteq\mathbb{I}}\varprojlim\left(\mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}(\mathbf{v}_J\mathcal{H},\mathbf{v}_K\mathcal{H})/\mathrm{Hom}_{\widehat{\mathcal{H}}}(\mathbf{v}_J\mathcal{H},\mathbf{v}_K\mathcal{I}_m)\right)\\
\cong& \varprojlim \mathcal{S}/\mathcal{J}_m = \widehat{\mathcal{S}}_{\mathbf{i}}.
\end{aligned}$$

The proposition follows. □

Recall the idempotent decomposition of $\widehat{\mathcal{H}}_{\mathbf{i}}$ from (3.8). We would next like to focus on the corresponding decomposition for $\widehat{\mathcal{S}}_{\mathbf{i}}$. Our notation follows the setup in [SW11] and [KL09].

5.2. **Idempotent decomposition.** Recall our fixed $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$. Let $J \subseteq \mathbb{I}$ and $\mathbf{u} = (u_1, \dots, u_n) \in \mathfrak{S}\mathbf{i}$. It will be convenient to encode the pair by splitting the tuple $\mathbf{u} = (u_1, \dots u_n)$ into blocks determined by J. More precisely, we write

$$(\mathbf{u}, J) \quad = \quad (u_1 \cdots u_{t_1} | u_{t_1+1} \cdots u_{t_2} | \cdots | u_{t_{r-1}+1} \cdots u_{t_r}) \tag{5.2}$$

where $t_r = n$, and a line is drawn between u_k and u_{k+1} if and only if $k \notin J$. In particular, in the extreme cases we have $(\mathbf{u}, \emptyset) = (u_1|u_2|\cdots|u_n)$ and $(\mathbf{u}, \mathbb{I}) = (u_1, u_2, \cdots, u_n)$.

For $(\mathbf{u}, J)$ we denote $\mathbf{u}_J = (\mathbf{u}', J)$, where $\mathbf{u}'$ is the unique element in the W_J-orbit of $\mathbf{u}$ where the integers in the parts between the lines in (5.2) are ascending, i.e.

$$u'_1 \leq \cdots \leq u'_{t_1}, \quad u'_{t_1+1} \leq \cdots \leq u'_{t_2}, \quad \dots \quad , \quad u'_{t_{r-1}+1} \leq \cdots \leq u'_n \tag{5.3}$$

Here, if q is an $e-th$ roots of unity, we order our chosen representatives $1, \dots, e$ for $\mathbb{Z}/e\mathbb{Z}$ as $1 < \cdots < e$. For $J \subseteq \mathbb{I}$, we denote by

$$\mathbf{U}_J = \{\mathbf{u}_J \mid \mathbf{u} \in \mathfrak{S}\mathbf{i}\}$$

the set of such representatives of W_J-orbits in $\mathfrak{S}\mathbf{i}$. Given $\mathbf{u} \in \mathfrak{S}\mathbf{i}$ and $J \subset K \subseteq \mathbb{I}$ we call $\mathbf{u}_J$ a *refinement* of $\mathbf{u}_K$.

Example 5.2. Let $n = 7$ and $J = \{1, 3, 5\} \subset K = \{1, 2, 3, 5, 6\}$. With $\mathbf{u} = (1, 2, 1, 1, 2, 1, 1)$ we have $\mathbf{u}_J = (1, 2|1, 1|1, 2|1)$ and $\mathbf{u}_K = (1, 1, 1, 2|1, 1, 2)$. Then $\mathbf{u}_J$ a refinement of $\mathbf{u}_K$. Note that indeed the additional vertical lines in $\mathbf{u}_J$ provide a refinement of the parts of $\mathbf{u}_K$.

Attached to $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$ we have the idempotent

$$\overset{+}{e}_{\mathbf{u}_J} \quad = \sum_{\mathbf{u}' \in \mathrm{W}_J \cdot \mathbf{u}} e_{\mathbf{u}'} \quad \in \widehat{\mathcal{H}}_{\mathbf{i}}. \tag{5.4}$$

These idempotents have the following important property.

Lemma 5.3. *For $J \subset \mathbb{I}$ and $\mathbf{u}_J \in \mathbf{U}_J$, the elements $\overset{+}{e}_{\mathbf{u}_J}, \mathbf{v}_J$ (in $\widehat{\mathcal{H}}_{\mathbf{i}}$) commute.*

Proof. We may assume $J \neq \emptyset$, since $\mathbf{v}_\emptyset = 1$ and the claim is obvious in this case. Suppose first that $\mathrm{W}_J \cong \mathfrak{S}_k$ for some k and consider the subalgebra $\widehat{\mathcal{H}}_{J,\mathbf{i}}$. Its identity element is $\overset{+}{e}_{\mathbf{u}_J}$ and we obtain $\widehat{\mathcal{H}}_{J,\mathbf{i}} = \overset{+}{e}_{\mathbf{u}_J}\widehat{\mathcal{H}}_{J,\mathbf{i}} = \widehat{\mathcal{H}}_{J,\mathbf{i}}\overset{+}{e}_{\mathbf{u}_J}$. In particular $\overset{+}{e}_{\mathbf{u}_J}$ commutes with $\mathbf{v}_J$. Note that in the extremal case $J = \mathbb{I}$, the element $\overset{+}{e}_{\mathbf{u}_\mathbb{I}}$ is just the identity element in $\widehat{\mathcal{H}}_{J,\mathbf{i}}$. Otherwise, we can find a proper decomposition $J = J_1 \cup J_2 \subset \mathbb{I}$ such that $W_J = \mathrm{W}_{J_1} \times \mathrm{W}_{J_2}$. Then $\overset{+}{e}_{\mathbf{u}_J} = \overset{+}{e}_{\mathbf{u}_{J_1}}\overset{+}{e}_{\mathbf{u}_{J_2}}$ and $\mathbf{v}_J = \mathbf{v}_{J_1}\mathbf{v}_{J_2}$. By definition, $\mathbf{v}_{J_i}$ commutes with $\overset{+}{e}_{\mathbf{u}_{J_j}}$ if $i \neq j$ and also, by the extremal case treated above, with $\overset{+}{e}_{\mathbf{u}_{J_i}}$. Hence the claim follows. $\square$

Lemma 5.3 directly implies

Corollary 5.4. *There is an isomorphism of algebras*

$$\begin{aligned}&\bigoplus_{J,K \subseteq \{s_1,\dots,s_{n-1}\}} \mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}(\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}}, \mathbf{v}_K\widehat{\mathcal{H}}_{\mathbf{i}}) \\ &\qquad\cong \bigoplus_{J,K \subseteq \{s_1,\dots,s_{n-1}\}} \bigoplus_{\mathbf{u}_J \in \mathbf{U}_J, \mathbf{u}'_K \in \mathbf{U}_K} \mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}(\overset{+}{e}_{\mathbf{u}_J}\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}}, \overset{+}{e}_{\mathbf{u}'_K}\mathbf{v}_K\widehat{\mathcal{H}}_{\mathbf{i}}).\end{aligned}$$

5.3. **Splits and merges in Vignéras' basis.** We now define certain split and merge maps motivated by the construction in [SW11].

Let $J \subset K \subseteq \mathbb{I}$, $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, which uniquely defines $\mathbf{u}_K \in \mathbf{U}_K$, of which $\mathbf{u}_J$ is a refinement. We use the notation from Section 5.2.

Definition 5.5. i.) We have $\overset{+}{e}_{\mathbf{u}_J}\mathbf{b}^1_{J,K}\overset{+}{e}_{\mathbf{u}_K} \neq 0$, and this is called a *split* of $\mathbf{u}_K$. If $|K \setminus J| = 1$, we call $J \subset K$ a *simple inclusion*, and $\overset{+}{e}_{\mathbf{u}_J}\mathbf{b}^1_{J,K}\overset{+}{e}_{\mathbf{u}_K}$ is called a *simple split*.

ii.) We have $\overset{+}{e}_{\mathbf{u}_K}\mathbf{b}^1_{K,J}\overset{+}{e}_{\mathbf{u}_J} \neq 0$, and this is called a *merge* of $\mathbf{u}_J$. Again, if $|K \setminus J| = 1$, it is called a *simple merge* of $\mathbf{u}_J$.

iii.) We denote by $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \in D^K_J$ the unique element with $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J}\mathbf{u}_J = \mathbf{u}_K$.

Note that any split (resp. merge) can be written as a sequence of simple splits (respectively simple merges).

Example 5.6. In the setup of Example 5.2 we have a (non-simple) split of $\mathbf{u}_K$. In this case $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J} = s_3s_2s_6$.

5.4. **Dimension matrix and dimension vectors.** To $(\mathbf{u}, J)$ as in (5.2) we now associate several combinatorial objects and groups of permutations.

Definition 5.7. The *dimension matrix* attached to $(\mathbf{u}, J)$ is the $e \times r$-matrix $\mathtt{D}(\mathbf{u}_J)$ with entries in $\mathbb{Z}_{\geq 0}$ defined as

$$\mathtt{D}(\mathbf{u}, J) = \left(d_i^j(\mathbf{u}, J)\right), \quad 1 \leq i \leq e, 1 \leq j \leq r, \tag{5.5}$$

where $\quad d_i^j = d_i^j(\mathbf{u}, J) = |\{k \mid t_{j-1} + 1 \leq k \leq t_j, u_k = i\}| \quad$ with $t_0 = 0$.

Note that d_i^j counts precisely the number of occurrences of i in the jth block of $(\mathbf{u}, J)$, whereas d_i is the total number of i, and d^j gives the size of the jth block. For $1 \leq i \leq e$ fixed set $\mathtt{D}_i = \mathtt{D}_i(\mathbf{u}, J) = (d_i^1(\mathbf{u}, J), \ldots, d_i^r(\mathbf{u}, J))$, and for fixed $1 \leq j \leq e$ set $\mathtt{D}^j = \mathtt{D}^j(\mathbf{u}, J) = (d_1^j(\mathbf{u}, J), \ldots, d_e^j(\mathbf{u}, J))$. The first encodes the multiplicities how often a certain number appears in each part, the second encodes for a fixed part the multiplicities of the numbers occurring in it.

Let $d_i = d_i(\mathbf{u}, J) = \sum_{j=1}^r d_i^j(\mathbf{u}, J)$ and $d^j = d^j(\mathbf{u}, J) = \sum_{i=1}^e d_i^j(\mathbf{u}_J)$. We call $\mathtt{d} = \mathtt{d}(\mathbf{u}, J) = (d_1, d_2, \ldots, d_e)$ the *dimension vector* and $\mathtt{t} = \mathtt{t}(\mathbf{u}, J) = (d^1, d^2, \ldots, d^r)$ the *type vector* attached to $(\mathbf{u}, J)$. Hence the dimension vector encodes the multiplicities how often each number occurs in total, whereas the type encodes the sizes of the parts ignoring which numbers occur. Note that the dimension vector only depends in $\mathbf{i}$ and we thus also write $\mathtt{d} = \mathtt{d}(\mathbf{i})$.

Example 5.8. In the setup of Example 5.2 the dimension matrix for $\mathbf{u}_J$ is given by $d_1^1 = d_1^3 = d_1^4 = 1$ and $d_1^2 = 2$, whereas $d_2^1 = d_2^3 = 1$ and $d_2^2 = d_2^4 = 0$. The dimension vector is $(5, 2)$ and its type vector is $(2, 2, 2, 1)$. On the other hand, for $\mathbf{u}_K$, we have for the values $d_1^1 = 3$ and $d_1^2 = 2$, and $d_2^1 = d_2^3 = 1$. The dimension vector is again $(5, 2)$, but the type vector is $(4, 3)$.

Given $(\mathbf{u}, J)$ we have now several (sub)groups of permutations attached to it (where we omit the $(\mathbf{u}, J)$ in the notation on the right hand side):

$$\mathfrak{S}_{\mathtt{D}_i(\mathbf{u},J)} = \mathfrak{S}_{d_i^1} \times \mathfrak{S}_{d_i^2} \cdots \times \mathfrak{S}_{d_i^r} < \mathfrak{S}_{d_i}, \tag{5.6}$$

$$\mathfrak{S}_{\mathtt{D}_\mathtt{d}(\mathbf{u},J)} = \mathfrak{S}_{\mathtt{D}_1} \times \cdots \times \mathfrak{S}_{\mathtt{D}_e} < \mathfrak{S}, \tag{5.7}$$

$$\mathfrak{S}_{\mathtt{D}^j(\mathbf{u},J)} = \mathfrak{S}_{d_1^j} \times \mathfrak{S}_{d_2^j} \cdots \times \mathfrak{S}_{d_e^j} < \mathfrak{S}_{d^j}, \tag{5.8}$$

$$\mathfrak{S}_{\mathtt{D}_\mathtt{t}(\mathbf{u},J)} = \mathfrak{S}_{\mathtt{D}^1} \times \cdots \times \mathfrak{S}_{\mathtt{D}^r} < \mathfrak{S}. \tag{5.9}$$

Note that choosing $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$ has the nice effect that $\mathfrak{S}_{\mathtt{D}_\mathtt{t}(\mathbf{u}_J)} = \mathrm{W}_J \cap \mathrm{Stab}_{\mathfrak{S}}\mathbf{u}$ is a standard parabolic subgroup. Note that $\mathfrak{S}_{\mathtt{D}_\mathtt{d}(\mathbf{u}_J)} \cong \mathfrak{S}_{\mathtt{D}_\mathtt{t}(\mathbf{u}_J)}$, since both groups precisely describe all permutations of $\mathbf{u}_J$ such that the number as well as the parts given by J are preserved.

Example 5.9. In the setup of Example 5.2 we have $\mathfrak{S}_{\mathtt{D}_\mathtt{d}(\mathbf{u}_J)} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_0 \times \mathfrak{S}_1 \times \mathfrak{S}_0$ and $\mathfrak{S}_{\mathtt{D}_\mathtt{t}(\mathbf{u}_J)} = \mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_0 \times \mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_1 \times \mathfrak{S}_0$.

In the following we will often drop the dependence on $(\mathbf{u}, J)$ in the notation, if we have some fixed $(\mathbf{u}, J)$, and we will only ever consider the case where $(\mathbf{u}, J) = \mathbf{u}_J$ for some $\mathbf{u}_J \in \mathbf{U}_J$. In this case, the groups (5.7) and (5.9)

and then also the groups (5.6) and (5.8) are generated by certain standard generators $s_i \in \mathfrak{S}$ labeled by a subset of $\mathbb{I}$. It will be convenient to use also different labellings of the generators which reflect directly the respective product decompositions.

Definition 5.10. For the group (5.7) the ath generator in the ith factor is denoted $s_{i,a}$, whereas for (5.9) the ath generator of the jth factor is denoted $s_a^{(j)}$.

In this notation we can make the above isomorphism explicit:

Lemma 5.11. *There is an isomorphism of groups*

$$\zeta_{\mathbf{u}_J}: \quad \mathfrak{S}_{\mathtt{D}_\mathtt{d}(\mathbf{u}_J)} \cong \mathfrak{S}_{\mathtt{D}_\mathtt{t}(\mathbf{u}_J)} \qquad s_{i,a} \mapsto s_l^{(t)}, \tag{5.10}$$

where t is such that $\sum_{k=1}^{t-1} d_i^k < a \leq \sum_{k=1}^{t} d_i^k$ and $l = a + \sum_{k=1}^{i-1} d_k^t$.

Proof. Since the two groups define the same subgroup of $\mathfrak{S}$, it suffices to compare their images there. But $s_{i,a}$ corresponds to $s_b \in \mathfrak{S}$, where $b = a + (\sum_{k=1}^{t-1} d^k) + (\sum_{k=1}^{i-1} d_k^t)$ whereas $s_l^{(t)}$ corresponds to $s_b \in \mathfrak{S}$, where $b = c + (\sum_{k=1}^{t-1} d^k)$. Hence the claim follows. □

Definition 5.12. In the following we will abbreviate the group in (5.10) by $\mathfrak{S}_{\mathbf{u}_J}$, but keep the two realisations in mind. Note that it is a standard parabolic subgroup of $\mathfrak{S}$ and we define $I_{\mathbf{u}_J}$ by $\mathrm{W}_{I_{\mathbf{u}_J}} = \mathfrak{S}_{\mathbf{u}_J}$. For $1 \leq i \leq e$ and $1 \leq a \leq d_i$ we abbreviate

$$(i,a)_{\mathbf{u}_J} \quad = \quad a + \left(\sum_{k=1}^{t-1} d^k\right) + \left(\sum_{k=1}^{i-1} d_k^t\right) \tag{5.11}$$

with t such that $\sum_{k=1}^{t-1} d_i^k < a \leq \sum_{k=1}^{t} d_i^k$.

Note that $(i,a)_{u_J}$ is just the position where the ath number i occurs in $\mathbf{u}$.

Example 5.13. Let us consider $\mathbf{u}_J = (1,1,2|1,1,1,2,2|1,1,2)$. Hence $n = 11, r = 3$ and $J = \{s_1, s_2, s_4, s_5, s_6, s_7, s_9, s_{10}\}$ with $\mathrm{W}_J \cong \mathfrak{S}_3 \times \mathfrak{S}_5 \times \mathfrak{S}_3$. In the usual generators of $\mathfrak{S}$ we have $\mathfrak{S}_{\mathtt{D}_1} \cong \langle s_1, s_4, s_5, s_9 \rangle \cong \mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_2$ and $\mathfrak{S}_{\mathtt{D}_2} = \langle s_7 \rangle \cong \mathfrak{S}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_1$ and then $\mathfrak{S}_{\mathtt{D}_\mathtt{d}} = \langle s_1, s_4, s_5, s_7, s_9 \rangle \cong \mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_2 \times \mathfrak{S}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_1$. It agrees with $\mathfrak{S}_{\mathtt{D}_\mathtt{t}}$ as a subgroup of $\mathfrak{S}$ (or W_J).

Now $\mathfrak{S}_{\mathtt{D}_\mathtt{d}} = \langle s_{1,1}, s_{1,3}, s_{1,4}, s_{1,6}, s_{2,2} \rangle$ and $\mathfrak{S}_{\mathtt{D}_\mathtt{t}(\mathbf{u}_J)} = \langle s_1^{(1)}, s_1^{(2)}, s_2^{(2)}, s_3^{(2)}, s_1^{(3)} \rangle$. The isomorphism ζ_{u_J} sends $s_{1,1}, s_{1,3}, s_{1,4}, s_{1,6}, s_{2,2}$ to $s_1^{(1)}, s_1^{(2)}, s_2^{(2)}, s_3^{(2), s_1^{(3)}}$ respectively. The corresponding elements $s_{(i,a)_{u_J}} \in \mathfrak{S}$ are s_1, s_4, s_5, s_9, s_7.

5.5. **Rings of invariants.** We now consider invariant polynomials for (5.7) and (5.9). Our different choices of generators come along with different labellings of the variables.

We attach, to our fixed $\mathbf{i}$, the following polynomial rings

$$R_+ = \Bbbk[Y_{1,1}, Y_{1,2}, \dots, Y_{1,d_1}, Y_{2,1}, \dots, Y_{2,d_2}, \dots Y_{e,d_e}]$$
$$= \Bbbk[Y_{1,1}, \dots, Y_{1,d_1}] \otimes_{\Bbbk} \Bbbk[Y_{2,1}, \dots, Y_{2,d_2}] \otimes_{\Bbbk} \dots \otimes_{\Bbbk} \Bbbk[Y_{e,1}, \dots, Y_{e,d_e}].$$

and

$$R_- = \Bbbk[Y_{1,1}^{-1}, Y_{1,2}^{-1}, \dots, Y_{1,\gamma_1}^{-1}, Y_{2,1}^{-1}, \dots, Y_{2,\gamma_2}^{-1}, \dots, Y_{e,\gamma_e}^{-1}]$$
$$= \Bbbk[Y_{1,1}^{-1}, \dots, Y_{1,d_1}^{-1}] \otimes_{\Bbbk} \Bbbk[Y_{2,1}^{-1}, \dots, Y_{2,d_2}^{-1}] \otimes_{\Bbbk} \dots \otimes_{\Bbbk} \Bbbk[Y_{e,1}^{-1}, \dots, Y_{e,d_e}^{-1},]$$

in n variables. For each $\mathbf{u}_J \in \mathbf{U}_J$ for some $J \subset \mathbb{I}$, we fix the isomorphisms of rings

$$\zeta_{+,\mathbf{u}_J} : R_+ \cong \Bbbk[X_1, \dots, X_n] \quad \text{and} \quad \zeta_{-,\mathbf{u}_J} : R_- \cong \Bbbk[X_1^{-1}, \dots, X_n^{-1}].$$

sending $Y_{i,a}^{\pm}$ to $X_{(i,a)_{\mathbf{u}_J}}^{\pm}$ with the notation from Definition 5.12. Together with Lemma 5.11 this also gives canonical identifications of invariants

$$R_+^{\mathfrak{S}_{\mathtt{D_d}(\mathbf{u}_J)}} = \Bbbk[X_1, \dots, X_n]^{\mathfrak{S}_{\mathtt{D_t}(\mathbf{u}_J)}}, \qquad R_-^{\mathfrak{S}_{\mathtt{D_d}(\mathbf{u}_J)}} = \Bbbk[X_1^{-1}, \dots, X_n^{-1}]^{\mathfrak{S}_{\mathtt{D_t}(\mathbf{u}_J)}}$$

Again, we will often abbreviate these invariants as $R_+^{\mathfrak{S}_{\mathbf{u}_J}} = \Bbbk[X_1, \dots, X_n]^{\mathfrak{S}_{\mathbf{u}_J}}$, respectively $R_-^{\mathfrak{S}_{\mathbf{u}_J}} = \Bbbk[X_1^{-1}, \dots, X_n^{-1}]^{\mathfrak{S}_{\mathbf{u}_J}}$.

Let $\hat{R}_+$ and $\hat{R}_-$ be the completions of R_+ and R_- at the maximal ideals generated by all the $Y_{i,j}$ respectively $Y_{i,j}^{-1}$. We have isomorphisms

$$\hat{\zeta}_{+,\mathbf{u}_J} : R_+ \cong \Bbbk[[X_1, \dots, X_n]] \quad \text{and} \quad \hat{\zeta}_{-,\mathbf{u}_J} : R_- \cong \Bbbk[[X_1^{-1}, \dots, X_n^{-1}]].$$

induced by $\zeta_{\pm,\mathbf{u}_J}$.

For $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, again with the notation from Definition 5.12, define the $\Bbbk$-linear inclusions

$$\begin{aligned} \xi_{\mathbf{u}} : \hat{R}_+ &\hookrightarrow \bigoplus_{w \in D^J_{\emptyset, I_{\mathbf{u}_J}}} \Bbbk[[X_1, \dots, X_n]]\, e_{w\cdot\mathbf{u}}, &(5.12)\\ Y_{i,a} &\mapsto \sum_{w \in D^J_{\emptyset, I_{\mathbf{u}_J}}} \left(X_{(i,a)_{w\cdot\mathbf{u}}}\, e_{w\cdot\mathbf{u}} \right), \\ \xi^{\sharp}_{\mathbf{u}} : \hat{R}_- &\hookrightarrow \bigoplus_{w \in D^J_{\emptyset, I_{\mathbf{u}_J}}} \Bbbk[[X_1^{-1}, \dots, X_n^{-1}]]\, e_{w\cdot\mathbf{u}}, &(5.13)\\ Y_{i,a}^{-1} &\mapsto \sum_{w \in D^J_{\emptyset, I_{\mathbf{u}_J}}} (X^{-1}_{(i,a)_{w\cdot\mathbf{u}}}\, e_{w\cdot\mathbf{u}}). \end{aligned}$$

Denote by $\hat{R}_{\pm,\mathbf{u}_J}$ the images of $\hat{R}_{\pm}^{\mathfrak{S}_{\mathbf{u}_J}}$ under $\xi_{\mathbf{u}}$ respectively $\xi^{\sharp}_{\mathbf{u}}$. In the following, we will identify elements with their images, i.e. we will view elements of $\hat{R}_{+,\mathbf{u}_J}$ alternatively as formal power series in the variables $Y_{i,a}$ or as formal power series of the form $\sum_{w \in D^J_{\emptyset, I_{\mathbf{u}_J}}} f_w e_{w\cdot\mathbf{u}}$ where each f_w is a formal power series in $X_1, \dots, X_n$, and similarly for elements of $\hat{R}_{-,\mathbf{u}_J}$.

5.6. **The completion of $\mathcal{Q}$.** The goal of this subsection is to describe the completion of the subalgebra $\mathcal{Q}$, which will play a similar role to the completion of the subalgebra $\mathcal{P}$ in $\mathcal{H}$ in giving rise to a completed faithful representation.

The *completion* $\widehat{\mathcal{Q}}_{\mathbf{i}}$ of $\mathcal{Q}$ is spanned by the elements in $\mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}(\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}}, \mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}})$, for $J \subseteq \mathbb{I}$, which are equal to left multiplication with an element $f \in \widehat{\mathcal{P}}$ such that $(T_i - q)f\overset{+}{e}_{\mathbf{u}_J}\mathbf{v}_J = 0$, see Remark 4.21. Equivalently, $\widehat{\mathcal{Q}}_{\mathbf{i}}$ is, via the decomposition $\widehat{\mathcal{P}}_{\mathbf{i}} \cong \bigoplus_{\mathbf{u}\in\mathfrak{S}\mathbf{i}} e_{\mathbf{u}}\widehat{\mathcal{P}}_{\mathbf{i}}$ and Lemma 5.3, spanned by those elements in $\mathrm{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}(\overset{+}{e}_{\mathbf{u}_J}\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}}, \overset{+}{e}_{\mathbf{u}_J}\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}})$ for some $J \subseteq \mathbb{I}$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, which are equal to left multiplication with some $f \in \bigoplus_{\mathbf{u}'\in \mathrm{W}_J\cdot\mathbf{u}} e_{\mathbf{u}'}\widehat{\mathcal{P}}$ such that $(T_i - q)f\overset{+}{e}_{\mathbf{u}_J}\mathbf{v}_J = 0$. This last property can be rephrased, similarly to Lemma 4.7, as follows.

Lemma 5.14. *Let $K \subseteq \mathbb{I}$ and $\mathbf{u}_K = (\mathbf{u}, K) \in U_K$. Furthermore, assume that $f \in \bigoplus_{w\in D^K_{\emptyset,I_{\mathbf{u}_K}}} \Bbbk[[X_1^{-1}, \dots, X_n^{-1}]]\, e_{w\cdot\mathbf{u}}$. Then*

$$f \in \hat{R}_{-,\mathbf{u}_K} \text{ if and only if } (T_i - q)f\overset{+}{e}_{\mathbf{u}_K}\mathbf{v}^{(K)} = 0 \text{ for all } i \in K.$$

Proof. Assume that $f \in \hat{R}_{-,\mathbf{u}_K}$ and pick $i \in K$. We have to show that $(T_i - q)f\sum_{\mathbf{u}'\in \mathrm{W}_K\cdot\mathbf{u}} e_{\mathbf{u}'}\mathbf{v}^{(K)} = 0$. It then suffices to verify, for any $w \in D^K_{\emptyset,K_{\mathbf{u}}}$

$$(T_i - q)f\,(e_{w\cdot\mathbf{u}} + e_{s_iw\cdot\mathbf{u}})\,\mathbf{v}^{(K)} = 0$$

if $s_iw\cdot\mathbf{u} \neq w\cdot\mathbf{u}$ and $(T_i - q)fe_{w\cdot\mathbf{u}}\mathbf{v}^{(K)} = 0$ otherwise. Assume first $s_iw\cdot\mathbf{u} \neq w\cdot\mathbf{u}$. The, setting $\beta_i = \frac{(q-1)}{1-X_iX_{i+1}^{-1}}$, we have the equalities

$$\begin{aligned}
& (T_i - q)Y_{c,a}^{-1}\,(e_{w\cdot\mathbf{u}} + e_{s_iw\cdot\mathbf{u}})\,\mathbf{v}^{(K)} \\
\overset{(5.13)}{=}\quad & (T_i - q)\left(X^{-1}_{(c,a)_{w\cdot\mathbf{u}}}e_{(w\cdot\mathbf{u})} + X^{-1}_{(c,a)_{s_iw\cdot\mathbf{u}}}e_{s_iw\cdot\mathbf{u}}\right)\mathbf{v}^{(K)} \\
=\quad & (T_i - q)\left(X^{-1}_{(c,a)_{w\cdot\mathbf{u}}}e_{(w\cdot\mathbf{u})} + X^{-1}_{s_i((c,a)_{w\cdot\mathbf{u}})}e_{s_iw\cdot\mathbf{u}}\right)\mathbf{v}^{(K)} \\
\overset{(3.10)}{=}\quad & (\Phi_i - q + \beta_i)\left(X^{-1}_{(c,a)_{w\cdot\mathbf{u}}}e_{w\cdot\mathbf{u}} + X^{-1}_{s_i((c,a)_{w\cdot\mathbf{u}})}e_{s_iw\cdot\mathbf{u}}\right)\mathbf{v}^{(K)} \\
=\quad & X^{-1}_{s_i((c,a)_{w\cdot\mathbf{u}})}e_{s_iw\cdot\mathbf{u}}\Phi_ie_{w\cdot\mathbf{u}} + X^{-1}_{(c,a)_{w\cdot\mathbf{u}}}e_{w\cdot\mathbf{u}}\Phi_ie_{s_iw\cdot\mathbf{u}} \\
& \qquad -(q-\beta_i)(X^{-1}_{(c,a)_{w\cdot\mathbf{u}}}e_{w\cdot\mathbf{u}} + X^{-1}_{s_i((c,a)_{w\cdot\mathbf{u}})}e_{s_iw\cdot\mathbf{u}}\mathbf{v}^{(K)} \\
=\quad & (X^{-1}_{s_i((c,a)_{w\cdot\mathbf{u}})}e_{s_iw\cdot\mathbf{u}} + X^{-1}_{(c,a)_{w\cdot\mathbf{u}}}e_{w\cdot\mathbf{u}})(\Phi_i - q + \beta_i)\,(e_{w\cdot\mathbf{u}} + e_{s_iw\cdot\mathbf{u}})\,\mathbf{v}^{(K)} \\
\overset{(3.10)}{=}\quad & (X^{-1}_{(c,a)_{s_iw\cdot\mathbf{u}}}e_{s_iw\cdot\mathbf{u}} + X^{-1}_{(c,a)_{w\cdot\mathbf{u}}}e_{w\cdot\mathbf{u}})(T_i - q)\mathbf{v}^{(K)} \\
\overset{(H\text{-}1)}{=}\quad & 0.
\end{aligned}$$

Let now $s_iw\cdot\mathbf{u} = w\cdot\mathbf{u}$. Then $(w^{-1}s_iw)\cdot\mathbf{u} = \mathbf{u}$, hence $t = w^{-1}s_iw \in \mathrm{W}_{\mathbf{u}_K}$. If we view $t \in \mathfrak{S}_{\mathtt{D}_{\mathtt{d}}(\mathbf{u}_J)}$ via Definition 5.12, then by assumption f is t-invariant. On the other hand, by our definitions, f being invariant under t when written in the

$Y_{c,a}^{-1}$ is equivalent to f being invariant under s_i when written in the X_j^{-1}. But now Lemma 4.7 implies $(T_i - q)fe_{w\cdot\mathbf{u}}\mathbf{v}^{(K)} = e_{w\cdot\mathbf{u}}(T_i - q)f\mathbf{v}^{(K)} = 0$. Hence the "if" part of the statement follows.

Now assume that $(T_i - q)f\overset{+}{e}_{\mathbf{u}_K}\mathbf{v}^{(K)} = 0$ for all $i \in K$. If $\mathfrak{S}_{\mathbf{u}_K}$ is trivial, then there is nothing to show. Otherwise let s_b be a standard generator in $\mathfrak{S}_{\mathbf{u}_K}$, in particular $s_b\mathbf{u} = \mathbf{u}$. Let $b = (c,a)_{\mathbf{u}}$. Then $s_b\mathbf{u} = \mathbf{u}$ implies that $(c, a+1)_{\mathbf{u}} = b + 1 = (c,a)_{\mathbf{u}} + 1$.

We can write $f = Y_{c,a+1}^{-1}g + h$ for some unique $g, h \in \hat{R}_-^{s_{c,a}} = \hat{R}_-^{s_b}$ and, noting that $\xi_{\mathbf{u}}^{\sharp}(g) \in \Bbbk[[X_1^{-1}, \ldots, X_n^{-1}]]^{s_b}e_{\mathbf{u}}\mathbf{v}$ for any $g \in \hat{R}_-^{s_{c,a}}$, we have

$$\begin{aligned} 0 &= e_{\mathbf{u}}(T_b - q)fe_{\mathbf{u}}\mathbf{v}^{(K)} \\ &= e_{\mathbf{u}}(T_b - q)Y_{c,a+1}^{-1}ge_{\mathbf{u}}\mathbf{v}^{(K)} + e_{\mathbf{u}}(T_b - q)he_{\mathbf{u}}\mathbf{v}^{(K)} \\ &= e_{\mathbf{u}}(T_b - q)X_{b+1}^{-1}ge_{\mathbf{u}}\mathbf{v}^{(K)} + 0 \quad (\text{since } h \in \hat{R}_-^{s_b}) \\ &\overset{\text{(H-7)}}{=} e_{\mathbf{u}}(qX_b^{-1}T_b^{-1} - qX_{b+1}^{-1})ge_{\mathbf{u}}\mathbf{v}^{(K)} \\ &\overset{(3.1)}{=} e_{\mathbf{u}}(X_b^{-1} - qX_{b+1}^{-1})ge_{\mathbf{u}}\mathbf{v}^{(K)}. \end{aligned}$$

The latter is nonzero in case $g \neq 0$. Hence $(T_j - q)f\overset{+}{e}_{\mathbf{u}_K}\mathbf{v}^{(K)} = 0$ implies $g = 0$ and therefore f has to be invariant under s_b, and then under all generators of $\mathfrak{S}_{\mathbf{u}_K}$. Thus $f \in \hat{R}_{-,\mathbf{u}_K}$. $\square$

For $J \subseteq \mathbb{I}$, denote by $\overrightarrow{\mathfrak{X}_J^{>}} \subset \overrightarrow{\mathfrak{X}_J}$ (respectively $\overleftarrow{\mathfrak{X}_J^{>}} \subset \overleftarrow{\mathfrak{X}_J}$) the subset of monomials $X_1^{a_1}\cdots X_n^{a_n}$ with $a_i \geq 0$ for all i, and by $\overrightarrow{\mathfrak{X}_J^{<}} \subset \overrightarrow{\mathfrak{X}_J}$ (respectively $\overleftarrow{\mathfrak{X}_J^{<}} \subset \overleftarrow{\mathfrak{X}_J}$) the subset of monomials $X_1^{a_1}\cdots X_n^{a_n}$ with $a_i \leq 0$ for all i. Note that $\flat$ from Section 2.1 induces a bijection between $\overrightarrow{\mathfrak{X}_J^{>}}$ and $\overleftarrow{\mathfrak{X}_J^{<}}$ and between $\overleftarrow{\mathfrak{X}_J^{>}}$ and $\overrightarrow{\mathfrak{X}_J^{<}}$.

As a consequence we obtain a topological basis of $\widehat{\mathcal{Q}}_{\mathbf{i}}$.

Proposition 5.15. *The completion $\widehat{\mathcal{Q}}_{\mathbf{i}}$ of $\mathcal{Q}$ has topological $\Bbbk$-basis given by*

$$\widehat{\mathcal{B}}_{\mathcal{Q}} = \{\hat{\mathbf{b}}_{J,J}^{p}\overset{+}{e}_{\mathbf{u}_J} \mid J \subseteq \mathbb{I}, \mathbf{u}_J \in \mathbf{U}_J, p \in \overleftarrow{\mathfrak{X}_{J_{\mathbf{u}}}^{<}}\} \tag{5.14}$$

where $\hat{\mathbf{b}}_{J,J}^{p}\overset{+}{e}_{\mathbf{u}_J} \in \operatorname{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}(\overset{+}{e}_{\mathbf{u}_J}\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}}, \overset{+}{e}_{\mathbf{u}_J}\mathbf{v}_J\widehat{\mathcal{H}}_{\mathbf{i}})$ is the homomorphism given by

$$\hat{\mathbf{b}}_{J,J}^{p}\overset{+}{e}_{\mathbf{u}_J}(\overset{+}{e}_{\mathbf{u}_J}\mathbf{v}_J) = \sum_{w \in D_{\emptyset, I_{\mathbf{u}_J}}^{J}} w(p)e_{w\cdot\mathbf{u}_J}\overset{+}{e}_{\mathbf{u}_J}\mathbf{v}_J.$$

Proof. The fact that this set spans $\widehat{\mathcal{Q}}_{\mathbf{i}}$ follows directly from the first paragraph of this subsection reformulated using the equivalence from Lemma 5.14. It is linearly independent as a direct consequence of (4.22). $\square$

5.7. **A basis of $\widehat{\mathcal{S}}_{\mathbf{i}}$.** The main goal of this subsection is the following basis theorem (with the notation from Proposition 5.15).

Proposition 5.16. *For fixed $K_1, K_2 \subseteq \mathbb{I}$, the set*

$$\widehat{\mathcal{B}}_{K_2,K_1} = \left\{ \mathbf{b}^1_{K_2,dJ}\mathbf{b}^d_{dJ,J}\hat{\mathbf{b}}^p_{J,J}\overset{+}{e}_{\mathbf{u}_J}\mathbf{b}^1_{J,K_1} \;\middle|\; \begin{array}{l} d \in D^{\mathbb{I}}_{K_2,K_1}, \\ J = K_1 \cap d^{-1}K_2, \\ \mathbf{u}_J \in \mathbf{U}_J, p \in \overrightarrow{\mathfrak{X}}^-_{I_{\mathbf{u}_J}} \end{array} \right\}$$

is a topological $\Bbbk$-basis for $\operatorname{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}(\mathbf{v}_{K_1}\widehat{\mathcal{H}}_{\mathbf{i}}, \mathbf{v}_{K_2}\widehat{\mathcal{H}}_{\mathbf{i}})$.

Proof. By Proposition 4.20 we have (with the notation from (5.1))

$$\mathcal{S}(\mathcal{Q} \cap \mathcal{J}_m)\mathcal{S} = \mathcal{S}(\mathcal{Q} \cap \mathcal{S}\mathbf{m}^m_\chi\mathcal{S})\mathcal{S} = \mathcal{S}(\mathcal{Q}\mathbf{m}^m_\chi\mathcal{Q})\mathcal{S} = \mathcal{S}\mathbf{m}^m_\chi\mathcal{S} = \mathcal{J}_m$$

where the third equality follows from $\mathbf{m}^m_\chi \subset Z(\mathcal{H}) \subset \mathcal{Q}$.

Now, as a $\mathcal{Q}$-module, we have $\mathcal{S} \cong \bigoplus_{\mathbf{x}\in\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}} \mathcal{Q} \circledast \mathbf{x}$ by Lemma 4.23 (with the notation defined there). Since $\mathbf{m}_\chi$ is central, the actions by left multiplication, right multiplication, or the $\circledast$-action induced by $\mathbf{m}_\chi \subset \mathcal{Q}$ all coincide, so

$$\mathcal{S}/\mathcal{J}_m = \mathcal{S}/\mathcal{S}\mathbf{m}^m_\chi \cong (\bigoplus_{\mathbf{x}\in\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}} \mathcal{Q} \circledast \mathbf{x})/(\bigoplus_{\mathbf{x}\in\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}} \mathcal{Q}\mathbf{m}^m_\chi \circledast \mathbf{x}) \cong \bigoplus_{\mathbf{x}\in\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}} (\mathcal{Q}/\mathcal{Q}\mathbf{m}^m_\chi) \circledast \mathbf{x}.$$

Thus we obtain

$$\begin{aligned} \widehat{\mathcal{S}}_{\mathbf{i}} &= \varprojlim \mathcal{S}/\mathcal{J}_m = \varprojlim \bigoplus_{\mathbf{x}\in\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}} (\mathcal{Q}/\mathcal{Q}\mathbf{m}^m_\chi) \circledast \mathbf{x} = \bigoplus_{\mathbf{x}\in\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}} \varprojlim (\mathcal{Q}/\mathcal{Q}\mathbf{m}^m_\chi) \circledast \mathbf{x} \\ &= \bigoplus_{\mathbf{x}\in\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}} \widehat{\mathcal{Q}}_{\mathbf{i}} \circledast \mathbf{x}. \end{aligned}$$

In particular $\widehat{\mathcal{S}}_{\mathbf{i}}$ is now free over $\widehat{\mathcal{Q}}_{\mathbf{i}}$ on basis $\mathcal{B}^{\mathcal{S}}_{\mathcal{Q}}$, which, together with Proposition 5.15 and the definition of $\circledast$ implies the desired basis for $\widehat{\mathcal{S}}_{\mathbf{i}}$. ☐

We have the following direct consequence.

Corollary 5.17. *Let $K_1, K_2 \subseteq \mathbb{I}$ and moreover let $\mathbf{u}'_{K_1} = (\mathbf{u}', K_1) \in U_{K_1}$ and $\mathbf{u}''_{K_2} = (\mathbf{u}'', K_2) \in U_{K_2}$. Then a basis of $\operatorname{Hom}_{\widehat{\mathcal{H}}_{\mathbf{i}}}(\overset{+}{e}_{\mathbf{u}'_{K_1}}\mathbf{v}_{K_1}\widehat{\mathcal{H}}_{\mathbf{i}}, \overset{+}{e}_{\mathbf{u}''_{K_2}}\mathbf{v}_{K_2}\widehat{\mathcal{H}}_{\mathbf{i}})$ is given by*

$$\left\{ \overset{+}{e}_{\mathbf{u}''_{K_2}}\mathbf{b}^1_{K_2,dJ}\mathbf{b}^d_{dJ,J}\hat{\mathbf{b}}^p_{J,J}\overset{+}{e}_{\mathbf{u}_J}\mathbf{b}^1_{J,K_1}\overset{+}{e}_{\mathbf{u}'_{K_1}} \;\middle|\; \begin{array}{l} d \in D^{\mathbb{I}}_{K_2,K_1}, \\ J = K_1 \cap d^{-1}K_2, \\ p \in \overrightarrow{\mathfrak{X}}^-_J, \\ \mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J \textit{ with} \\ \mathbf{u}'_{K_1} = \mathbf{u}_{K_1}, \\ \mathbf{u}''_{K_2} = (d\cdot\mathbf{u})_{K_2} \end{array} \right\} \tag{5.15}$$

where we note that $\mathbf{u}'_{K_1} = \mathbf{u}_{K_1}$ means $\mathbf{u}' \in \mathrm{W}_{K_1}\cdot\mathbf{u}$ and similarly $\mathbf{u}''_{K_2} = (d\cdot\mathbf{u})_{K_2}$ means $\mathbf{u}'' \in \mathrm{W}_{K_2}\cdot(d\cdot\mathbf{u})$.

5.8. **A faithful representation of $\widehat{\mathcal{S}}_{\mathbf{i}}$.** In this subsection we will construct a faithful representation of $\widehat{\mathcal{S}}_{\mathbf{i}}$ similar to the construction in Theorem 4.8 .

For $K \subseteq \mathbb{I}$ and $\mathbf{u}_K \in \mathbf{U}_K$, define

$$\widehat{\mathbb{P}(\mathcal{S})}_{\mathbf{i}}^{\mathbf{u}_K} = \hat{R}_{-,\mathbf{u}_K}\mathbf{v}^{(K)} \quad \text{and} \quad \widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}^{\mathbf{u}_K} = \hat{R}_{+,\mathbf{u}_K}\overline{\mathbf{v}}^{(K)}$$

where again, as in Theorem 4.8, the superscript in $\overline{\mathbf{v}}^{(K)}$ is just a formal index. Moreover set

$$\widehat{\mathbb{P}(\mathcal{S})}_{\mathbf{i}} = \bigoplus_{K\subseteq\mathbb{I}}\bigoplus_{\mathbf{u}_K\in\mathbf{U}_K}\widehat{\mathbb{P}(\mathcal{S})}_{\mathbf{i}}^{\mathbf{u}_K} \quad \text{and} \quad \widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}} = \bigoplus_{K\subseteq\mathbb{I}}\bigoplus_{\mathbf{u}_K\in\mathbf{U}_K}\widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}^{\mathbf{u}_K}. \tag{5.16}$$

These are the underlying spaces for two faithful representations:

Proposition 5.18. *i.) There is a faithful representation $\widehat{\rho}$ of $\widehat{\mathcal{S}}_{\mathbf{i}}$ on $\widehat{\mathbb{P}(\mathcal{S})}_{\mathbf{i}}$ where the basis elements of $\widehat{\mathcal{S}}_{\mathbf{i}}$ as in* (5.15) *act via*

$$\begin{aligned}&\widehat{\rho}\left(\overset{+}{e}_{(w\cdot\mathbf{u})_{K_2}}\mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\hat{\mathbf{b}}^p_{J,J}\overset{+}{e}_{\mathbf{u}_J}\mathbf{b}^1_{J,K_1}\overset{+}{e}_{\mathbf{u}_{K_1}}\right)f\mathbf{v}^{(K_1)}\\ = \quad &\overset{+}{e}_{(w\cdot\mathbf{u})_{K_2}}\left(\sum\nolimits_{a\in D^{K_2}_{\emptyset,wJ}}T_a\right)T_w g_p\overset{+}{e}_{\mathbf{u}_J}\left(\sum\nolimits_{b\in D^{K_1}_{\emptyset,J}}T_b\right)f\mathbf{v}^{(K_2)}\end{aligned}$$

for $f\mathbf{v}^{(K_1)} \in \widehat{\mathbb{P}(\mathcal{S})}_{\mathbf{i}}^{\mathbf{u}_{K_1}}$, where g_p is as defined in (4.24) *and $\mathbf{u}_K = (\mathbf{u}, K)$.*

ii.) There is a faithful representation $\widehat{\overline{\rho}}$ of $\widehat{\mathcal{S}}_{\mathbf{i}}$ on $\widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}$ where basis elements of $\widehat{\mathcal{S}}_{\mathbf{i}}$ as in (5.15) *act via*

$$\begin{aligned}&\widehat{\overline{\rho}}\left(\overset{+}{e}_{(w\cdot\mathbf{u})_{K_2}}\mathbf{b}^1_{K_2,wJ}\mathbf{b}^w_{wJ,J}\hat{\mathbf{b}}^p_{J,J}\overset{+}{e}_{\mathbf{u}_J}\mathbf{b}^1_{J,K_1}\overset{+}{e}_{\mathbf{u}_{K_1}}\right)f\overline{\mathbf{v}}^{(K_1)}\\ = &\overset{+}{e}_{(w\cdot\mathbf{u})_{K_2}}\left(\sum\nolimits_{a\in D^{K_2}_{\emptyset,wJ}}T^\sharp_a\right)T^\sharp_w g^\sharp_p\overset{+}{e}_{\mathbf{u}_J}\left(\sum\nolimits_{b\in D^{K_1}_{\emptyset,J}}T^\sharp_b\right)f\overline{\mathbf{v}}^{(K_2)}.\end{aligned}$$

for $f\overline{\mathbf{v}}^{(K_1)} \in \widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}^{\mathbf{u}_{K_1}}$, where g_p is as defined in (4.24) *and $\mathbf{u}_K = (\mathbf{u}, K)$.*

Proof. We prove (i), the proof of (ii) being analogous. We first claim that

$$\widehat{\mathbb{P}(\mathcal{S})}_{\mathbf{i}} \cong \widehat{\mathcal{S}}_{\mathbf{i}} \otimes_{\mathcal{S}} \mathbb{P}(\mathcal{S}).$$

Indeed, using that

$$\mathbb{P}(\mathcal{S})^K = \left\{ f\mathbf{v}^{(K)} \;\middle|\; \begin{array}{l} f\mathbf{v}^{(K)} \in \Bbbk[X_1^{\pm1},\dots,X_n^{\pm1}]\mathbf{v}^{(K)} \\ (T_i - q)f\mathbf{v}^{(K)} = 0 \text{ for all } i \in K \end{array} \right\},$$

we see that

$$\begin{aligned}\overset{+}{e}_{\mathbf{u}_K}\widehat{\mathcal{S}}_{\mathbf{i}} \otimes_{\mathcal{S}} \mathbb{P}(\mathcal{S})^K \quad &= \quad \left\{ f\mathbf{v}^{(K)} \;\middle|\; \begin{array}{l} f\mathbf{v}^{(K)} \in \bigoplus_{\mathbf{u}'\in \mathrm{W}_K\cdot\mathbf{u}_K} e_{\mathbf{u}'}\Bbbk[[X_1^{-1},\dots,X_n^{-1}]]\mathbf{v}^{(K)} \\ (T_i - q)f\mathbf{v}^{(K)} = 0 \text{ for all } i \in K \end{array} \right\}\\ &= \quad \widehat{\mathbb{P}(\mathcal{S})}_{\mathbf{i}}^{\mathbf{u}_K}\end{aligned}$$

by Lemma 5.14; hence the claim follows. Since we defined our action $\widehat{\rho}$ to coincide with ρ (cf. (4.6) and Remark 4.18) on elements of $\mathcal{S}$, the fact that $\widehat{\rho}$ is a faithful representation follows immediately from Theorem 4.8. $\square$

Remark 5.19. Similarly to Corollary 3.14, we have an isomorphism of $\widehat{\mathcal{S}}_{\mathbf{i}}$-modules

$$\widehat{\mathbb{P}(\mathcal{S})}_{\mathbf{i}} \cong {}^{\sharp}(\widehat{\overline{\mathbb{P}}(\mathcal{S})}_{-\mathbf{i}}) \quad \text{via} \quad f\overset{+}{e}_{\mathbf{u}_{K_1}}\mathbf{v}^{(K_1)} \mapsto f^{\sharp}\overset{+}{e}_{-\mathbf{u}_{K_1}}\overline{\mathbf{v}}^{(K_1)}.$$

6. The action of (algebraic) merges on $\widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}$

In this section, we describe, explicitly and in detail, the action of a simple merge on the twisted faithful representation $\widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}$, as we will later use this to compare $\mathcal{S}_{\mathbf{i}}$ to the quiver Schur algebra. In Proposition 6.19, we will deduce a generating set for the Schur algebra which refines Corollary 4.13.

6.1. **Basic formulae for algebraic merges.** We start by describing some combinatorics of distinguished coset representatives in $D_{\emptyset,J}$, where $J \subseteq \mathbb{I}$.

Thus let $J \subseteq \mathbb{I}$ be fixed. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and consider $(\mathbf{u}, J)$ as in (5.2). Then for a permutation $w \in \mathfrak{S}$ we have that

$$w \in D_{\emptyset,J} \quad \Leftrightarrow \quad w(k_j+1) < w(k_j+2) < \cdots < w(k_j+d^j), \quad \text{for } 1 \leq j \leq r,$$

where $k_j = \sum_{j' \leq j} d^{j'}$ (and $d^0 = 0$). That means the numbers inside each part of J are kept increasing when applying the permutation $w \in \mathfrak{S}$. When drawing the corresponding permutation diagram, this means that two strands from the same part given by J do not cross.

Lemma 6.1. *Let $J = I - \{a\}$ for some $a \in \mathbb{I}$ and set $b = n - a$. Then $D_{\emptyset,J}$ consists precisely of the elements*

$$(s_{c_b}s_{c_b+1}\cdots s_{n-1})(s_{c_{b-1}}\ldots s_{n-2})\cdots(s_{c_2}\ldots s_{a+1})(s_{c_1}s_{c_1+1}\ldots s_a), \qquad (6.1)$$

where $1 \leq c_1 < c_2 < \cdots < c_b$ and by convention $(s_r s_{r+1}\cdots s_k) = 1$ if $r > k$.

Proof. This is a standard fact, for a proof see for e.g. [Str05, Proposition A.2]. □

Definition 6.2. For $J \subseteq K \subseteq \mathbb{I}$, define the *algebraic merges*

$$\pitchfork_J^K = \sum_{w \in D_{\emptyset,J}^K} w, \quad \text{and} \quad {}_q\pitchfork_J^K = \sum_{w \in D_{\emptyset,J}^K} T_w^{\sharp}. \qquad (6.2)$$

We also write $\pitchfork_{\mathtt{t}(J)}^{\mathtt{t}(K)}$ instead of $\pitchfork_J^K$, respectively ${}_q\pitchfork_{\mathtt{t}(J)}^{\mathtt{t}(K)}$ instead of ${}_q\pitchfork_J^K$.

Remark 6.3. Note that $\pitchfork_J^K = \pitchfork_{J,c}^{K,c}$ and ${}_q\pitchfork_J^K = {}_q\pitchfork_{J,c}^{K,c}$ for any c if we interpret the smaller symmetric group as a subgroup of the larger symmetric group. We will use this fact tacitly in the following. Moreover, by definition and using (H-7) and Remark 3.5, we have $\pitchfork_J^K f = f \pitchfork_J^K$ and ${}_q\pitchfork_J^K f = f {}_q\pitchfork_J^K$ for any W_J-invariant polynomial.

Example 6.4. For instance, if $J = \mathbb{I} - \{a\}$ for some $a \in \mathbb{I}$ and $K = \mathbb{I}$,then $\pitchfork_J^K = \pitchfork_{a,n-a}^n$ is precisely the sum over all elements of the form (6.1).

We first state a few easy properties of these algebraic merges.

Lemma 6.5 (Associativity). *Assume $n = 1 + (a-1) + (b-1)$ with each summand in $\mathbb{Z}_{\geq 0}$. Then*

$$\pitchfork^{a+b-1}_{a,b-1}\pitchfork^{a,b-1}_{a-1,1,b-1} \quad = \quad \pitchfork^{a+b-1}_{a-1,1,b-1} \quad = \quad \pitchfork^{a+b-1}_{a-1,b}\pitchfork^{a-1,b}_{a-1,1,b-1}\,. \tag{6.3}$$

The analogous formula holds for the ${}_q\pitchfork$ as well.

Proof. The middle term of (6.3) is precisely the sum of all permutations w of $1, 2, \ldots, n$ such that the numbers stay increasing inside the parts of size $a-1, 1, b-1$. But each such w can be written as a composition of the form $w = xy$, where $y \in \mathfrak{S}_a$ permutes the first a numbers, but keeps the first $a-1$ increasing, followed by a permutation x which keeps the first a numbers $y(i)$, $1 \leq i \leq n$ in order and keeps the last $b-1$ numbers increasing. Moreover, each such xy gives rise to a unique w in the sum. This proves the first equality. The second is similar starting with permuting the last b numbers instead. The statement for ${}_q\pitchfork$ follows by the same arguments. □

Lemma 6.6 (Splitting off a simple reflection). *Let $n = a + b$ with $a, b \in \mathbb{Z}_{>0}$. Then we have the following equalities*

$$\begin{aligned}
\pitchfork^{a+b}_{a,b} \quad &= \quad \pitchfork^{1,a+b-1}_{1,a,b-1}\, s_1 \cdots s_a + \pitchfork^{1,a+b-1}_{1,a-1,b} && (6.4)\\
&= \quad \pitchfork^{1,a+b-1}_{1,a,b-1}\pitchfork^{a+1,b-1}_{a,1,b-1} - \pitchfork^{1,a+b-1}_{1,a-1,b}\pitchfork^{a,b}_{a,1,b-1} + \pitchfork^{1,a+b-1}_{1,a-1,b}\,. && (6.5)
\end{aligned}$$

The analogous formulae hold for the ${}_q\pitchfork$ as well.

Proof. Consider the set of elements from (6.1) and divide them into those which contain s_1 (in the rightmost factor) and those which do not. This division corresponds precisely to the two summands on the right hand side of (6.4). Note that as a special case of (6.4) we obtain

$$\pitchfork^{a+1}_{a,1} \quad = \quad \pitchfork^{1,a}_{1,a}\, s_1 s_2 \cdots s_a + \pitchfork^{1,a}_{1,a-1,1} = s_1 s_2 \cdots s_a + \pitchfork^{1,a}_{1,a-1,1}\,. \tag{6.6}$$

To verify (6.5), it suffices to show

$$\pitchfork^{1,a+b-1}_{1,a,b-1}\pitchfork^{a+1,b-1}_{a,1,b-1} - \pitchfork^{1,a+b-1}_{1,a-1,b}\pitchfork^{a,b}_{a,1,b-1} \quad = \quad \pitchfork^{1,a+b-1}_{1,a,b-1}\, s_1 \cdots s_a. \tag{6.7}$$

However, thanks to (6.6), the left hand side equals

$$\begin{aligned}
LHS \quad &= \quad \pitchfork^{1,a+b-1}_{1,a,b-1}\left(s_1 \cdots s_a + \pitchfork^{1,a,\quad b-1}_{1,a-1,1,b-1}\right) - \pitchfork^{1,a+b-1}_{1,a-1,b}\pitchfork^{a,\quad b}_{a,1,b-1}\\
&= \quad \pitchfork^{1,a+b-1}_{1,a,b-1}\, s_1 \cdots s_a + \pitchfork^{1,a+b-1}_{1,a-1,b}\pitchfork^{1,a-1,b}_{1,a-1,1,b-1} - \pitchfork^{1,a+b-1}_{1,a-1,b}\pitchfork^{a,b}_{a,1,b-1}\\
&= \quad \pitchfork^{1,a+b-1}_{1,a,b-1}\, s_1 \cdots s_a + \pitchfork^{1,a+b-1}_{1,a-1,b}\left(\pitchfork^{1,a-1,b}_{1,a-1,1,b-1} - \pitchfork^{a,b}_{a,1,b-1}\right)\\
&= \quad \pitchfork^{1,a+b-1}_{1,a,b-1}\, s_1 \cdots s_a.
\end{aligned}$$

Here the second equality follows from (6.3), the third is clear and the last one follows from the obvious fact that the expression inside the brackets is zero. Hence the claim follows. Note that the same arguments work for ${}_q\pitchfork$ as well, since we have not used the quadratic relation (H-1). □

Definition 6.7. For $1 \leq i \neq j \leq n$ set

$$\beta_{i,j} = qX_i - X_j \quad \text{and} \quad \gamma_{i,j} = X_i - X_j \quad \text{and finally} \quad \theta_{i,j} = \frac{\beta_{ij}}{\gamma_{ij}}. \tag{6.8}$$

Lemma 6.8. *The equality $\theta_{1,2}\theta_{2,3} - \theta_{1,2}\theta_{1,3} + \theta_{1,3}\theta_{3,2} = q$ holds.*

Proof. One easily checks that $\beta_{2,3}\gamma_{1,3} - \beta_{1,3}\gamma_{2,3} = (q-1)X_3\gamma_{1,2}$. Thus, if we set $\gamma = \gamma_{2,3}\gamma_{1,3}$, then

$$\theta_{1,2}(\theta_{2,3} - \theta_{1,3}) \quad = \quad \frac{\beta_{1,2}}{\gamma_{1,2}}\left(\frac{\beta_{2,3}}{\gamma_{2,3}} - \frac{\beta_{1,3}}{\gamma_{1,3}}\right) \quad = \quad \frac{(q-1)X_3\beta_{1,2}}{\gamma}. \tag{6.9}$$

On the other hand one checks easily that $q\gamma_{1,3}\gamma 2,3 + \beta_{1,3}\beta_{3,2} = (q-1)\beta_{1,2}$ and thus

$$q - \theta_{1,3}\theta_{3,2} \quad = \quad \frac{q\gamma_{1,3}\gamma_{2,3} + \beta_{1,3}\beta_{3,2}}{\gamma} \quad = \quad \frac{(q-1)X_3\beta_{1,2}}{\gamma}. \tag{6.10}$$

Subtracting (6.10) from (6.9) gives $\theta_{1,2}(\theta_{2,3} - \theta_{1,3}) - q + \theta_{1,3}\theta_{3,2} = 0$. □

The action of simple merges on polynomials and rational functions in the X_i is quite subtle, but produces interesting formulae.

Example 6.9. For instance we have $\pitchfork^{2}_{1,1}\,(\theta_{1,2}) = (1+q)$. This is because $\pitchfork^{2}_{1,1}\,(\theta_{1,2}) = (1+s_1)(\theta_{1,2}) = \frac{qX_1 - X_2 - qX_2 + X_1}{X_1 - X_2} = (1+q)$.

More generally, we have the following equalities of rational functions in the X_i:

Lemma 6.10. *Let $1 \leq c \leq n-1$ and $0 \leq a \leq n$. Then the following holds*

i.) For any a: $\pitchfork^{a,c}_{a,1,c-1}\,(\prod_{k=a+2}^{a+c} \theta_{a+1,k}) = \sum_{k=0}^{c-1} q^r$.
ii.) For $a \geq 1$: $\pitchfork^{a,c}_{a,1,c-1}\,(\theta_{1,a+1}\prod_{k=a+2}^{a+c} \theta_{a+1,k}) = \prod_{k=a+1}^{a+c} \theta_{1,k} + \sum_{r=1}^{c-1} q^r$.

The same formulae hold for the ${}_q\pitchfork$ as well.

Proof. Without loss of generality, we may assume $a = 0$ in (i) and $a = 1$ in (ii), since the general case then follows by just shifting the labels. We prove both statements in parallel, using induction on c. The base case is $c = 2$. (For the extreme case $c = 1$ we have $\pitchfork^{1,1}_{1,1}\,(\theta_{1,2})$ respectively $\pitchfork^{1}_{1,0}\,(1) = 1$ by convention.) For (i), the base case is Example 6.9, whereas for (ii) we need to show $(1+s_2)(\theta_{1,2}\theta_{2,3}) = \theta_{1,2}\theta_{1,3} + q$, or equivalently $\theta_{1,2}(\theta_{2,3} - \theta_{1,3}) = q - \theta_{1,3}\theta_{3,2}$. This, however, is Lemma 6.8. So assume now both, (i) and (ii), are true for $c - 1$.

For (i) we abbreviate $\Pi = \theta_{1,2}\prod_{k=3}^{c+1}\theta_{2,k} = \theta_{1,2}\Pi'$ and obtain

$$\begin{array}{rcl}
\pitchfork^{c}_{1,c-1}\,\Pi & \overset{(6.4)}{=} & \left(\pitchfork^{1,c-1}_{1,1,c-2}\,(1+s_1)\right)\Pi \\
& = & \Pi + \pitchfork^{1,c-1}_{1,1,c-2}\,\left(s_1(\theta_{1,2})\Pi'\right) \\
& = & \Pi + \pitchfork^{1,c-1}_{1,1,c-2}\,(1+q-\theta_{1,2})\Pi' \\
& = & \Pi + (1+q)\left(\pitchfork^{1,c-1}_{1,1,c-2}\,\Pi'\right) - \pitchfork^{1,c-1}_{1,1,c-2}\,\theta_{1,2}\Pi' \\
& \overset{\text{ind. hyp.}}{=} & \displaystyle\prod_{k=2}^{c}\theta_{1,k} + (1+q)\sum_{r=0}^{c-2}q^r - \prod_{k=2}^{c}\theta_{1,k} - \sum_{r=1}^{c-2}q^r \\
& = & \displaystyle\sum_{r=0}^{c-1}q^r,
\end{array}$$

where in the penultimate line we have used the induction hypothesis for $c-1$, namely (ii) for the first summand and (i) for the second summand. In the third line we have also used the induction hypothesis for $c=2$ for (ii).

For (ii), we abbreviate $Z = \prod_{k=2}^{c}\theta_{2,k} = \theta_{2,3}Z'$ and obtain

$$\begin{array}{rl}
& \pitchfork^{1,c}_{1,1,c-1}\,(\theta_{1,2}Z) \\
\overset{(6.4)}{=} & \left((\pitchfork^{1,1,c-1}_{1,1,1,c-2}\,s_2)+1\right)(\theta_{1,2}Z) \\
= & \theta_{1,2}Z + \pitchfork^{1,1,c-1}_{1,1,1,c-2}\,\left(s_2(\theta_{1,2}\theta_{2,3})Z'\right) \\
= & \theta_{1,2}Z + \pitchfork^{1,1,c-1}_{1,1,1,c-2}\,(\theta_{1,2}\theta_{1,3} - \theta_{1,2}\theta_{2,3} + q)Z' \\
= & \theta_{1,2}Z + \left(\theta_{1,2}\,\pitchfork^{1,1,c-1}_{1,1,1,c-2}\,\theta_{1,3}Z'\right) - \left(\theta_{1,2}\,\pitchfork^{1,1,c-1}_{1,1,1,c-2}\,\theta_{2,3}Z'\right) + q\left(\pitchfork^{1,1,c-1}_{1,1,1,c-2}\,Z'\right).
\end{array}$$

Now we use the induction hypothesis for $c-1$, namely (ii) for the middle and (i) for the last summand and obtain

$$(\theta_{1,2}\prod_{k=3}^{c+1}\theta_{2,k}) + (\prod_{k=2}^{c+1}\theta_{1,k}) + (\theta_{1,2}\sum_{r=1}^{c-2}q^r) - (\theta_{1,2}\prod_{k=3}^{c+1}\theta_{2,k}) - (\theta_{1,2}\sum_{r=1}^{c-2}q^r) + q\sum_{r=0}^{c-2}q^r$$

Hence, altogether we have

$$\pitchfork^{1,c}_{1,1,c-1}\,(\theta_{1,2}Z) \quad = \quad \prod_{k=2}^{c+1}\theta_{1,k} + q\sum_{r=0}^{c-2}q^r \quad = \quad \prod_{k=2}^{c+1}\theta_{1,k} + \sum_{r=1}^{c-1}q^r$$

This completes the proof. □

6.2. **Algebraic merges in the faithful representation.** In this subsection we give explicit formulae for the action of the simple merges on the faithful representation $\widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}$ from (5.16).

Setup for the whole subsection: Assume that $K = \mathbb{I}$, so $\mathbf{u}_K = (\mathbf{u}, K)$ with $\mathbf{u} = (1^{d_1}, 2^{d_2}, \ldots, e^{d_e})$ and $J = \mathbb{I}\setminus\{a\}$. Set $b = n-a$. Let $\mathbf{u}_J \in \mathbf{U}_J$. Set $a_i = d_i^1(\mathbf{u}_J)$ and $b_i = d_i^2(\mathbf{u}_J)$ in the notation of Definition 5.7. In particular,

$a_i + b_i = d_i$ for all $i = 1, \ldots, e$. Then the action of the merge from (4.2) can be expressed in the $\#$-twisted versions as follows:

Proposition 6.11. *For* $\widehat{\overline{\mathbb{P}}(\mathcal{S})_{\mathbf{i}}}$ *and* $f \in \hat{R}_{+,\mathbf{u}_J}$ *as in Proposition 5.18* (ii) *we have*

$$e_{\mathbf{u}}\widehat{\overline{\rho}}(\mathbf{b}^1_{K,J})f\overline{\mathbf{v}}^{(J)} \quad = \quad e_{\mathbf{u}_K} \pitchfork^{d_1, \quad d_2, \quad \ldots, d_e}_{a_1,b_1,a_2,b_2,\ldots,a_e,b_e} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \prod_{k=a+1}^{n} \prod_{l=1}^{a} \theta_{l,k} f\overline{\mathbf{v}}^{(K)}.$$

where $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J}$ *is as in Definition 5.5, explicitly*

$$\sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \quad = \quad \sigma^{(1^{d_1},2^{d_2},\ldots,e^{d_e})}_{(1^{a_1},2^{a_2},\ldots,e^{a_e}|1^{b_1},\ldots,e^{b_e})}. \tag{6.11}$$

Remark 6.12. Note that by Lemma 5.14 the component at $e_{\mathbf{u}}$ completely determines the element in $\widehat{\overline{\mathbb{P}}(\mathcal{S})_{\mathbf{i}}}$.

As a direct consequence from the definitions, (6.2), we obtain

Corollary 6.13. *We have*

$$e_{\mathbf{u}}\widehat{\overline{\rho}}(\mathbf{b}^1_{K,J})f\overline{\mathbf{v}}^{(J)} = e_{\mathbf{u}}\left(\sum_{w} w\sigma^{\mathbf{u}_K}_{\mathbf{u}_J}\right) \prod_{k=a+1}^{n} \prod_{l=1}^{a} \theta_{l,k} f\overline{\mathbf{v}}^{(K)}.$$

where the sum runs over $w \in D^{I_{\mathbf{u}_K}}_{\emptyset,\sigma^{\mathbf{u}_K}_{\mathbf{u}_J}(I_{\mathbf{u}_J})}$ *with notation from Definition 5.12.*

The proof of Proposition 6.11 is rather technical and occupies the rest of the subsection, proceeding by induction on $a + b$, with the base case being trivial. We start with some preparations.

Remark 6.14. Note that $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J}$ in (6.11) factorizes as $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J} = \sigma_1\sigma_2$ where

$$\begin{aligned}
\sigma_1 &= \sigma^{(1|1^{d_1-1},2^{d_2},3^{d_3}, \quad \ldots \quad ,e^{d_e})}_{(1|1^{a_1},\ldots,e^{a_e}|1^{b_1-1},2^{b_2},\ldots,e^{b_e})} \\
\sigma_2 &= \sigma^{(1^{a_1}|1,2^{a_2},\ldots\, e^{a_e}|1^{b_1-1},2^{b_2},\ldots,e^{b_e})}_{(1^{a_1}|2^{a_2},\ldots,e^{a_e}|1|1^{b_1-1},2^{b_2},\ldots,e^{b_e})} \quad = \quad s_{a_1+1}\cdots s_a
\end{aligned}$$

Moreover, we have $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J} = \sigma_3\sigma_4\sigma_5^{-1}$, where

$$\begin{aligned}
\sigma_3 &= \sigma^{(1|1^{d_1-1},\, 2^{d_2},\, 3^{d_3}, \quad \ldots \quad \ldots,\ t^{d_t}, \quad \ldots \quad \ldots\ e^{a_e})}_{(1|1^{a_1-1},2^{a_2},\ldots,t^{a_t+1},\ldots,e^{a_e}|1^{b_1},\ldots t^{b_t-1},\ \ldots,\ e^{b_e})} \\
\sigma_4 &= \sigma^{(1^{a_1},\ldots,t^{a_t+1},\ldots,e^{a_e}|1^{b_1},\ldots,t^{b_t-1},\ldots,e^{b_e})}_{(1^{a_1},2^{a_2},\ \ldots\ldots,\ e^{a_e}|t|1^{b_1},\ldots,t^{b_t-1},\ldots,e^{b_e})} = s_{a_1+a_2+\cdots+a_t+1}\cdots s_{a-1}s_a, \\
\sigma_5 &= \sigma^{(1^{a_1}\ldots e^{a_e}|1^{b_1} \quad \ldots \quad e^{b_e})}_{(1^{a_1}\ldots e^{a_e}|t|1^{b_1}\ldots t^{b_t-1}\ldots e^{b_e})} = s_{(a+\sum_{j=1}^{t-1} b_j)}\cdots s_{a+2}s_{a+1},
\end{aligned}$$

By definition of the representation in Proposition 5.18 (ii) we have

$$e_{\mathbf{u}}\widehat{\overline{\rho}}(\mathbf{b}^1_{K,J})f\overline{\mathbf{v}}^{(J)} \quad = \quad e_{\mathbf{u}}\ {}_q\pitchfork^{a+b}_{a,b} f\overline{\mathbf{v}}^{(J)} \quad =: \quad \boxed{?}$$

Applying Lemma 6.6 to the right hand side we obtain

$$\boxed{?} = e_{\mathbf{u}}\big({}_q\pitchfork^{1,a+b-1}_{1,\,a,\,b-1}\ {}_q\pitchfork^{a+1,b-1}_{a,\,1,\,b-1} + {}_q\pitchfork^{1,a+b-1}_{1,\,a-1,\,b}\left(1 - {}_q\pitchfork^{a,b}_{a,1,b-1}\right) f\overline{\mathbf{v}}^{(J)}.$$

Note that ${}_q\pitchfork^{a,b}_{a,1,b-1}$ commutes past f by Remark 6.3, and therefore we have ${}_q\pitchfork^{a,b}_{a,1,b-1} f\overline{\mathbf{v}} = \sum_{s=0}^{b-1} q^s f\overline{\mathbf{v}}$, using Lemma 6.1, (3.6) and $T_i\overline{\mathbf{v}} = -\overline{\mathbf{v}}$ by (3.1). Altogether we obtain

$$\boxed{?} \quad = \quad e_{\mathbf{u}} \left({}_q\pitchfork^{1,a+b-1}_{1,a,b-1} \, {}_q\pitchfork^{a+1,b-1}_{a,\,1,\,b-1} - {}_q\pitchfork^{1,a+b-1}_{1,\,a-1,\,b} \left(\sum_{s=1}^{b-1} q^s \right) \right) f\overline{\mathbf{v}}^{(J)}. \tag{6.12}$$

We can then rewrite the two summands, which we denote by $\boxed{?_1}$ and $\boxed{?_2}$ respectively,as in the following two lemmas.

Lemma 6.15. *The second summand in* (6.12) *equals*

$$\boxed{?_2} \quad = \quad - \left(\sum_{s=1}^{b-1} q^s \right) \pitchfork^{1,d_1-1,\quad d_2,\quad \ldots,d_e}_{1,a_1-1,b_1,a_2,b_2,\ldots,a_e,b_e} \, \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \left(\prod_{k=a+1}^{n} \prod_{j=2}^{a} \theta_{j,k} \right) f\overline{\mathbf{v}}^{(K)}.$$

where $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J}$ *is as in* (6.11).

Proof. First we analyse which idempotents e can appear to the right of the merge for the result not to be annihilated by $e_{\mathbf{u}}$, i.e.

$${}_q\pitchfork^{1,a+b-1}_{1,\,a-1,\,b} e \left(\sum_{s=1}^{b-1} q^s \right) f\overline{\mathbf{v}}^{(J)} \neq 0.$$

Clearly, any such e must be a summand of $\overset{+}{e}_{(1|1^{a_1-1},2^{a_2},\ldots,e^{a_e}|1^{b_1},\ldots,e^{a_e})}$. Moreover, note that $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J} = \sigma^{(1|1^{d_1-1},2^{d_2},\ldots,e^{d_e})}_{(1|1^{a_1-1},2^{a_2},\ldots,e^{a_e}|1^{b_1},\ldots,e^{b_e})}$. Then

$$e_{\mathbf{u}q}\pitchfork^{1,a+b-1}_{1,a-1,b} f\overline{\mathbf{v}}^{(J)} = \pitchfork^{1,d_1-1,\quad d_2,\quad \ldots,d_e}_{1,a_1-1,b_1,a_2,b_2,\ldots,a_e,b_e} \, \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} f\overline{\mathbf{v}}^{(J)}$$

by the inductive hypothesis, since $a+b-1 < a+b$. □

Lemma 6.16. *The first summand* $\boxed{?_1}$ *in* (6.12) *equals*

$$e_{\mathbf{u}} \Big[\pitchfork^{1,d_1-1,\quad d_2,\quad \ldots,d_e}_{1,a_1,b_1-1,a_2,b_2,\ldots,a_e,b_e} \, \sigma_1 P \, \pitchfork^{a_1+1,b_1-1,a_2,b_2,\ldots,a_e,b_e}_{a_1,\,1,\,b_1-1,a_2,b_2,\ldots,a_e,b_e} \, \sigma_2 \prod_{i=1}^{a} \theta_{i,a+1} \Big] f\overline{\mathbf{v}}^{(K)}$$
$$+ \quad e_{\mathbf{u}} \Big[\sum_{t=2}^{e} \pitchfork^{1,d_1-1,\quad d_2,\quad \ldots,d_t,\qquad\qquad d_e}_{1,a_1-1,b_1,a_2,b_2,\ldots,a_t+1,b_t-1\ldots,a_e,b_e} \, Z \prod_{i=1}^{a} \theta_{i,a+1} \Big] f\overline{\mathbf{v}}^{(K)}$$

where $P = \prod_{k=a+2}^{n} \prod_{j=2}^{a+1} \theta_{j,k}$, *and* $Z = \sigma_3 P \theta_{j,k} \, \pitchfork^{a_1,\ldots,a_t+1,a_{t+1},\ldots,a_e,b-1}_{a_1,\ldots,a_t,1,a_{t+1},\ldots,a_e,b-1} \, \sigma_4$. *The elements* σ *are as in Remark 6.14.*

Proof. To have $e_{\mathbf{u}q}\pitchfork^{1,a+b-1}_{1,a,b-1} \overset{+}{e}_{\mathbf{u}_{1,a,b-1}} \neq 0$ we need $\mathbf{u}_{1,a,b-1}$ to be of the form $(t|\ldots)$ for some $t \in \{1, 2, \ldots, e\}$. We distinguish two cases, namely

$$e_{\mathbf{u}} \, {}_q\pitchfork^{1,a+b-1}_{1,a,b-1} \, \overset{+}{e}_{(1|1^{a_1}\ldots e^{a_e}|1^{b_1-1}2^{b_2}\ldots e^{b_e})} \, {}_q\pitchfork^{a+1,b-1}_{a,1,b-1} \, \overset{+}{e}_{(1^{a_1},\ldots,e^{a_e}|1|1^{b_1-1},2^{b_2},\ldots,e^{b_e})}$$

and

$$e_{\mathbf{u}} \, {}_q\pitchfork^{1,a+b-1}_{1,a,b-1} \, \overset{+}{e}_{(t|1^{a_1}\ldots e^{a_e}|1^{b_1}\ldots t^{b_t-1}\ldots e^{b_e})} \, {}_q\pitchfork^{a+1,b-1}_{a,1,b-1} \, \overset{+}{e}_{(1^{a_1},\ldots,e^{a_e}|t|1^{b_1},\ldots,t^{b_t-1},\ldots,e^{b_e})}.$$

for $t = 2, \dots, e$. Then the claim follows directly from the definition of the permutations σ in Remark 6.14 and the induction hypothesis. $\square$

Lemma 6.17. *The second summand in Lemma 6.16, which we denote by* $\boxed{?_{1,2}}$*, equals*

$$e_{\mathbf{u}} \sum_{t=2}^{e} \pitchfork^{1,d_1-1,\ \ d_2,\dots,\ \ d_t,\ \ \ \ d_{t+1},\ \ d_e}_{1,a_1-1,b_1,a_2,b_2,\dots,a_t,1,b_t-1,a_{t+1}\dots,a_e,b_e}\ \sigma_3\sigma_4 P \prod_{i=1}^{a} \theta_{i,a+1} f \overline{\mathbf{v}}^{(K)}. \tag{6.13}$$

Proof. We first rewrite the term Z appearing in Lemma 6.16 as

$$\begin{aligned}\sigma_3 P \pitchfork^{a_1,\dots,a_t+1,a_{t+1},\dots,a_e,b-1}_{a_1,\dots,a_t,\ 1,\ a_{t+1},\dots,a_e,b-1} \sigma_4 \quad &= \quad \sigma_3 \pitchfork^{a_1,\dots,a_t+1,a_{t+1},\dots,a_e,b-1}_{a_1,\dots,a_t,\ 1,\ a_{t+1},\dots,a_e,b-1} \sigma_4 P \\ &= \quad \pitchfork^{a_1,b_1,\dots,a_t+1,b_t-1,a_{t+1},b_{t+1},\dots,a_e,b_e}_{a_1,b_1,\dots,a_t,1,b_t-1,a_{t+1},b_{t+1},\ \dots,\ a_e,b_e} \sigma_3\sigma_4 P\end{aligned}$$

where for the first equality we have used that $P = \prod_{k=a+2}^{n}\prod_{j=2}^{a+1}\theta_{j,k}$ is σ_4-invariant and Remark 6.3. For the second equality one checks that σ_3 from Remark 6.14 commutes with $\pitchfork^{a_1,\dots,a_t+1,a_{t+1},\dots,a_e,b-1}_{a_1,\dots,a_t,\ 1,\ a_{t+1},\dots,a_e,b-1}$. Then the claim follows by substituting this into the formula from Lemma 6.16 and using the associativity of merges. $\square$

Lemma 6.18. *The first summand in Lemma 6.16, which we denote by* $\boxed{?_{1,1}}$*, equals*

$$\begin{aligned}\boxed{?_{1,1}} \quad = \quad & e_{\mathbf{u}}\Big[\pitchfork^{1,d_1-1,d_2,\ \ \dots,d_e}_{1,d_1-1,a_2,b_2,\dots,a_e,b_e} \pitchfork^{1,d_1-1,\ \ \ \ \ \ \ \ a_2,b_2,\dots,a_e,b_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\dots,a_e,b_e}\ \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} P \\ + \quad & \pitchfork^{1,d_1-1,\ \ d_2,\ \ \dots,d_e}_{1,a_1,b_1-1,a_2,b_2,\dots,a_e,b_e} (s_1 s_2 \cdots s_{a_1} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J}) \prod_{k=a+2}^{n}\prod_{j=1}^{a}\theta_{j,k}\Big] \prod_{i=1}^{a}\theta_{i,a+1} f \overline{\mathbf{v}}^{(K)}\end{aligned}$$

We first use the special case (6.6) of Lemma 6.6 to write

$$\pitchfork^{a_1+1,b_1-1,a_2,b_2,\dots,a_e,b_e}_{a_1,\ 1,\ b_1-1,a_2,b_2,\dots,a_e,b_e} \quad = \quad \pitchfork^{1,a_1,\ \ \ \ \ b_1-1,a_2,b_2,\dots,a_e,b_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\dots,a_e,b_e} + s_1 \cdots s_{a_1} \tag{6.14}$$

This element obviously commutes with $P = \prod_{k=a+2}^{n}\prod_{j=2}^{a+1}\theta_{j,k}$ by (H-7). The same holds for σ_2. Hence $\boxed{?_{1,1}}$ equals

$$e_{\mathbf{u}}\Big[\pitchfork^{1,d_1-1,\ \ d_2,\ \ \dots,d_e}_{1,a_1,b_1-1,a_2,b_2,\dots,a_e,b_e}\ \sigma_1 P \pitchfork^{a_1+1,b_1-1,a_2,b_2,\dots,a_e,b_e}_{a_1,\ 1,\ b_1-1,a_2,b_2,\dots,a_e,b_e}\ \sigma_2\Big] w$$

$$= \quad e_{\mathbf{u}} \pitchfork^{1,d_1-1,\ \ d_2,\ \ \dots,d_e}_{1,a_1,b_1-1,a_2,b_2,\dots,a_e,b_e}\ \sigma_1 \Big[\pitchfork^{1,a_1,\ \ \ \ \ b_1-1,a_2,b_2,\dots,a_e,b_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\dots,a_e,b_e}\ \sigma_2 P + P s_1 \cdots s_a\Big] w$$

$$= \quad e_{\mathbf{u}}\Big[\pitchfork^{1,d_1-1,\ \ d_2,\ \ \dots,d_e}_{1,a_1,b_1-1,a_2,b_2,\dots,a_e,b_e}\ \sigma_1 \pitchfork^{1,a_1,\ \ \ \ \ b_1-1,a_2,b_2,\dots,a_e,b_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\dots,a_e,b_e}\ \sigma_2 P \tag{6.15}$$

$$+ \pitchfork^{1,d_1-1,\ \ d_2,\ \ \dots,d_e}_{1,a_1,b_1-1,a_2,b_2,\dots,a_e,b_e}\ \sigma_1 s_1 \cdots s_a \prod_{k=a+2}^{n}\prod_{j=1}^{a}\theta_{j,k}\Big] w \tag{6.16}$$

where we have abbreviated $w = \prod_{i=1}^{a}\theta_{i,a+1} f \overline{\mathbf{v}}^{(K)}$, and, for the last equality, used that $s_1 \cdots s_a$ maps the set $\{1, 2, \dots, a\}$ to $\{2, 3, \dots, a+1\}$.

Next, observe that σ_1 and $\pitchfork^{1,a_1, \quad b_1-1,a_2,b_2,\dots,a_e,b_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\dots,a_e,b_e}$ commute. Therefore,

$$
\begin{aligned}
& \pitchfork^{1,d_1-1, \quad d_2, \quad \dots,d_e}_{1,a_1,b_1-1,a_2,b_2,\dots,a_e,b_e} \sigma_1 \pitchfork^{1,a_1, \quad b_1-1,a_2,b_2,\dots,a_e,b_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\dots,a_e,b_e} \sigma_2 \\
= \; & \pitchfork^{1,d_1-1, \quad d_2, \quad \dots,d_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\dots,a_e,b_e} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \\
= \; & \pitchfork^{1,d_1-1,d_2, \quad \dots,d_e}_{1,d_1-1,a_2,b_2,\dots,a_e,b_e} \pitchfork^{1,d_1-1, \quad a_2,b_2,\dots,a_e,b_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\dots,a_e,b_e} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J}
\end{aligned} \tag{6.17}
$$

by the associativity property for merges and the factorisation from Remark 6.14 for the first equality and again associativity for the last equality. Hence the claim follows by substituting the result (6.17) into (6.16) and, furthermore, using the equality $\sigma_1 s_1 \cdots s_a = s_1 s_2 \cdots s_{a_1} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J}$.

Let us summarise what we have so far. The left hand side of the asserted formula in Proposition 6.11 equals

$$
\begin{aligned}
e_{\mathbf{u}} \widehat{\widehat{\rho}}(\mathbf{b}^1_{K,J}) f \overline{\mathbf{v}}^{(J)} &= \boxed{?_{1,1}} + \boxed{?_{1,2}} + \boxed{?_2} \\
&= e_{\mathbf{u}} \pitchfork^{1,d_1-1,d_2, \quad \dots,d_e}_{1,d_1-1,a_2,b_2,\dots,a_e,b_e} Y \prod_{k=a+1}^{n} \prod_{j=2}^{a} \theta_{j,k} f \overline{\mathbf{v}}^{(K)},
\end{aligned} \tag{6.18}
$$

where Y is given by

$$
\pitchfork^{1,d_1-1, \quad a_2,b_2,\dots,a_e,b_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\dots,a_e,b_e} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \theta_{1,a+1} \prod_{k=a+2}^{a} \theta_{a+1,k} \tag{6.19}
$$

$$
+ \quad \pitchfork^{1,d_1-1, \quad a_2,b_2,\dots,a_e,b_e}_{1,a_1,b_1-1,a_2,b_2,\dots,a_e,b_e} \sigma_1 s_1 \cdots s_a \prod_{k=a+1}^{n} \theta_{1,k} \tag{6.20}
$$

$$
+ \quad \sum_{t=2}^{e} \pitchfork^{1,d_1-1, \quad a_2,b_2,\dots, \quad a_e,b_e}_{1,a_1-1,b_1,a_2,b_2,\dots,a_t,1,b_t-1,a_{t+1}\dots,a_e,b_e} \sigma_3 \sigma_4 \theta_{1,a+1} \prod_{k=a+2}^{a} \theta_{a+1,k} \tag{6.21}
$$

$$
- \quad \left(\sum_{s=1}^{b-1} q^s \right) \pitchfork^{1,d_1-1, \quad a_2,b_2,\dots,a_e,b_e}_{1,a_1-1,b_1,a_2,b_2,\dots,a_e,b_e} . \tag{6.22}
$$

Note that Y is just what is left over after we pull out the common parts of all three summands.

We now rewrite (6.21). We have

$$\begin{aligned}
&\sum_{t=2}^{e} \pitchfork^{1,d_1-1,\quad a_2,b_2,\ldots,\qquad\qquad\qquad a_e,b_e}_{1,a_1-1,b_1,a_2,b_2,\ldots,a_t,1,b_t-1,a_{t+1}\ldots,a_e,b_e} \sigma_3\sigma_4 \\
= \;& \pitchfork^{1,d_1-1}_{1,a_1-1,b_1} \sum_{t=2}^{e} \pitchfork^{1,a_1-1,b_1,a_2,b_2,\ldots,a_t,b_t,\ldots,a_e,b_e}_{1,a_1-1,b_1,a_2,b_2,\ldots,a_t,1,b_t-1,\ldots,a_e,b_e} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J}\sigma_5 \\
= \;& \pitchfork^{1,d_1-1}_{1,a_1-1,b_1} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \sum_{t=2}^{e} \pitchfork^{a+b_1+\cdots b_{t-1},b_t,b-b_1-\cdots-b_t}_{a+b_1+\cdots b_{t-1},1,b_t-1,b-(b_1+\cdots+b_t)} \sigma_5 \\
= \;& \pitchfork^{1,d_1-1}_{1,a_1-1,b_1} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \sum_{t=2}^{e}\sum_{l=0}^{b_t-1} s_{a+\sum_{j=1}^{t-1} b_j+l}\cdots s_{a+\sum_{j=1}^{t-1} b_j}\cdots s_{a+2}s_{a+1} \\
= \;& \pitchfork^{1,d_1-1}_{1,a_1-1,b_1} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \sum_{l=b_1}^{b-1} s_l s_{l-1}\cdots s_{a+1} \\
= \;& \pitchfork^{1,d_1-1}_{1,a_1-1,b_1} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} (\pitchfork^{a,b}_{a,1,b-1} - \pitchfork^{a,b_1,b-b_1}_{a,1,b_1-1,b-b_1}),
\end{aligned} \tag{6.23}$$

where for the first equality we have used Remark 6.14 and rewritten the merge as a product of two (non-interacting) merges, for the second equality the commutativity of $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J}$ with the respective merges, for the third equality Lemma 6.1 and the definition of σ_5, and finally for the last equality formula (6.6).

On the other hand, by Lemma 6.10 we have

$$\pitchfork^{a,b}_{a,1,b-1} \theta_{1,a+1}\left(\prod_{k=a+2}^{n}\theta_{a+1,k}\right) - \left(\sum_{s=1}^{b-1} q^s\right) = \prod_{k=a+1}^{n}\theta_{1,k}. \tag{6.24}$$

which simplifies (6.21)+(6.22) further. Altogether we obtain

$$Y = \pitchfork^{1,d_1-1,\qquad\quad a_2,b_2,\ldots,a_e,b_e}_{1,a_1-1,1,b_1-1,a_2,b_2,\ldots,a_e,b_e} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J}\theta_{1,a+1}\prod_{k=a+2}^{n}\theta_{a+1,k} \tag{6.25}$$

$$+ \pitchfork^{1,d_1-1,\quad a_2,b_2,\ldots,a_e,b_e}_{1,a_1,b_1-1,a_2,b_2,\ldots,a_e,b_e} s_1s_2\cdots s_{a_1}\sigma^{\mathbf{u}_K}_{\mathbf{u}_J}\prod_{k=a+1}^{n}\theta_{1,k} \tag{6.26}$$

$$+ \pitchfork^{1,d_1-1,\quad a_2,b_2,\ldots,a_e,b_e}_{1,a_1-1,b_1,a_2,b_2,\ldots,a_e,b_e} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J}\prod_{k=a+1}^{n}\theta_{1,k} \tag{6.27}$$

$$- \pitchfork^{1,d_1-1}_{1,a_1-1,b_1} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \pitchfork^{a,b_1,b-b_1}_{a,1,b_1-1,b-b_1} \theta_{1,a+1}\prod_{k=a+2}^{n}\theta_{a+1,k}. \tag{6.28}$$

Since $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \pitchfork^{a,b_1,b-b_1}_{a,1,b_1-1,b-b_1} = \pitchfork^{a_1,b_1}_{a_1,1,b_1-1} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} = \pitchfork^{1,a_1-1,b_1}_{1,a_1-1,1,b_1-1} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J}$, the terms (6.25) and (6.28) cancel each other. Applying Lemma 6.6 to (6.26)+(6.27) gives

$$Y = \pitchfork^{d_1,\quad a_2,b_2,\ldots,a_e,b_e}_{a_1,b_1,a_2,b_2,\ldots,a_e,b_e} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J}\prod_{k=a+1}^{n}\theta_{1,k}. \tag{6.29}$$

Substituting this back into (6.18), we obtain

$$\begin{aligned} & e_{\mathbf{u}_K}\widehat{\overline{\rho}}(\mathbf{b}^1_{K,J})f\overline{\mathbf{v}}^{(J)} \\ = \;& e_{\mathbf{u}_K} \pitchfork^{d_1-1,d_2,\quad\ldots,d_e}_{d_1,a_2,b_2,\ldots,a_e,b_e} \pitchfork^{d_1,\quad a_2,b_2,\ldots,a_e,b_e}_{a_1,b_1,a_2,b_2,\ldots,a_e,b_e} \sigma^{\mathbf{u}_K}_{\mathbf{u}_J} \prod_{k=a+1}^{n}\prod_{j=1}^{a}\theta_{j,k}f\overline{\mathbf{v}}^{(K)} \end{aligned}$$

The associativity property of merges gives finally the desired formula from Proposition 6.11. This finishes the proof of Proposition 6.11.

6.3. **A refined generating set of the affine Schur algebra $\mathcal{S}$.** In this section, we improve on our generating set for the algebra $\mathcal{S}$ from Corollary 4.13.

Proposition 6.19. *The algebra $\mathcal{S}$ is generated by*

$$\{\mathbf{b}^1_{K_2,K_1}, \mathbf{b}^p_{J,J} \mid K_1, K_2 \subseteq \mathbb{I}, p \in \overrightarrow{\mathfrak{X}}_{I_{\mathbf{u}_J}}\}. \tag{6.30}$$

In other words, the algebra is generated by the subalgebra $\mathcal{Q}$ from Proposition 4.20 and the splits and merges $\mathbf{b}^1_{K_1,K_2}$.

Proof. By Proposition 4.17 the proposed generating set together with $\mathbf{b}^d_{dJ,J}$, where $d \in D^{\mathbb{I}}_{K_2,K_1}$, $J = K_1 \cap d^{-1}K_2$ generate the algebra $\mathcal{S}$. Hence it suffices to show that the $\mathbf{b}^d_{dJ,J}$ are redundant. First note that $d \in D_{dJ,J}$, since for any $i \in J$ we have $l(ds_id^{-1}d) = l(ds_i) > l(d_i)$, because $J \subset K_1$ and $d \in D_{K_2,K_1}$. Therefore d permutes the blocks in J without changing the order inside the blocks. Moreover, $d \in D_{dJ,J}$ implies that $\mathbf{b}^d_{dJ,J}(\mathbf{v}_J) = \mathbf{v}_{dJ}T_d$ by (4.15). This also implies that without loss of generality we may assume that d only swaps two neighbouring parts of J, as an arbitrary permutation of parts can be written as a composition of swapping neighbouring ones. Since the arguments are all local we can even assume that J contains only two parts, i.e. $J = \mathbb{I}\backslash a$. Set $b = n-a$. Note that in this case $d \in D_{dJ,J}$ is then the shortest coset representative of the longest element in $\mathfrak{S}$. Hence $T^{\#}_d$ is the summand corresponding to the longest element $d = d^{a+b}_{a,b}$ appearing in $\pitchfork^{a+b}_{a,b}$. Define

$$\overline{{}_q\pitchfork^{a+b}_{a,b}} = {}_q\pitchfork^{a+b}_{a,b} - T^{\#}_d. \tag{6.31}$$

By Example 6.4 we have $({}_q\pitchfork^{a+b}_{a,b})^{\#}\mathbf{v}_J = ({}_q\pitchfork^{a+b}_{a,b}\overline{\mathbf{v}}_J)^{\#} = b^1_{\mathbb{I},J}(\mathbf{v}_J)$. Hence it suffices to show that $\overline{{}_q\pitchfork^{a+b}_{a,b}}^{\#}\mathbf{v}_J$ can be expressed in terms of simple splits and merges applied to $\mathbf{v}_J$. We argue by induction on $a+b=n$. The base case $a=b=1$, so $n=2$, is obvious. For the general case we first rewrite (6.31) using (6.5), namely

$$\overline{{}_q\pitchfork^{a+b}_{a,b}} = {}_q\pitchfork^{1,a+b-1}_{1,a,b-1}\,{}_q\pitchfork^{a+1,b-1}_{a,1,b-1} - {}_q\pitchfork^{1,a+b-1}_{1,a-1,b}\,{}_q\pitchfork^{a,b}_{a,1,b-1} - T^{\#}_d + {}_q\pitchfork^{1,a+b-1}_{1,a-1,b}. \tag{6.32}$$

On the other hand if we abbreviate $d_1 = d^{1,a+b-1}_{1,a,b-1}$ and $d_2 = d^{a+1,b-1}_{a,1,b-1}$ and set $D_1 = T^{\#}_{d_1}$ and $D_2 = T^{\#}_{d_2}$, then we obtain

$$\begin{aligned}
&{}_q\pitchfork^{1,a+b-1}_{1,a,b-1}\,{}_q\pitchfork^{a+1,b-1}_{a,1,b-1}\\
&= (\overline{{}_q\pitchfork^{1,a+b-1}_{1,a,b-1}} + D_1)(\overline{{}_q\pitchfork^{a+1,b-1}_{a,1,b-1}} + D_2)\\
&= \overline{{}_q\pitchfork^{1,a+b-1}_{1,a,b-1}}\;\overline{{}_q\pitchfork^{a+1,b-1}_{a,1,b-1}} + D_1\overline{{}_q\pitchfork^{a+1,b-1}_{a,1,b-1}} + \overline{{}_q\pitchfork^{1,a+b-1}_{1,a,b-1}}D_2 + D_1D_2\\
&= {}_q\pitchfork^{1,a+b-1}_{1,a,b-1}\;\overline{{}_q\pitchfork^{a+1,b-1}_{a,1,b-1}} + \overline{{}_q\pitchfork^{1,a+b-1}_{1,a,b-1}}\;{}_q\pitchfork^{a+1,b-1}_{a,1,b-1} - \overline{{}_q\pitchfork^{1,a+b-1}_{1,a,b-1}}\;\overline{{}_q\pitchfork^{a+1,b-1}_{a,1,b-1}} + D_1D_2
\end{aligned}$$

where for the last line we used the equalities $D_1 = \pitchfork^{1,a+b-1}_{1,a,b-1} - \overline{\pitchfork^{1,a+b-1}_{1,a,b-1}}$ and the analogous one for D_2.

By Lemma 6.1 we have $d_1 = (s_{b-1}s_b\cdots s_{n-1})(s_3s_4\cdots s_{a+1})(s_2s_3\cdots s_a)$ and $d_2 = s_1s_2\cdots s_{a-1}$, in particular $d = d_1d_2$ with $l(d) = l(d_1) + l(d_2)$. But this implies then $T^{\#}_d = D_1D_2$ and therefore we obtain from (6.32) and (6.3) the following equality

$$\begin{aligned}
\overline{{}_q\pitchfork^{a+b}_{a,b}} \quad = \quad & {}_q\pitchfork^{1,a+b-1}_{1,a,b-1}\;\overline{{}_q\pitchfork^{a+1,b-1}_{a,1,b-1}} + \overline{{}_q\pitchfork^{1,a+b-1}_{1,a,b-1}}\;{}_q\pitchfork^{a+1,b-1}_{a,1,b-1} - \overline{{}_q\pitchfork^{1,a+b-1}_{1,a,b-1}}\;\overline{{}_q\pitchfork^{a+1,b-1}_{a,1,b-1}}\\
& - {}_q\pitchfork^{1,a+b-1}_{1,a-1,b}\;{}_q\pitchfork^{a,b}_{a,1,b-1} + {}_q\pitchfork^{1,a+b-1}_{1,a-1,b}.
\end{aligned}$$

Applying $^{\#}$ to the whole equation and using the inductive hypothesis, the right hand side of the equation is in the subalgebra generated by our proposed generating set, hence so is the left hand side as desired. $\square$

7. Quiver Hecke algebras and the isomorphism theorem

In this section, we finally connect the constructions developed so far with the so-called quiver Hecke algebras originally introduced by Khovanov-Lauda [KL09] using diagrammatics and by Rouquier [Rou08] using algebraic and categorical constructions, and later connected to flagged quiver representations in [VV11]. The quiver Schur algebra treated in the next section is a generalisation of the quiver Hecke algebra introduced in [SW11] using flagged quiver representations where, generalising [VV11], partial flags are used instead of full flags only.

7.1. **The quiver Hecke algebra.** We identify the fixed representatives $1, \ldots, e$ of $\mathbb{Z}/e\mathbb{Z}$ with the vertices in the affine Dynkin diagram $\Gamma = \Gamma_e$ attached to the affine Kac-Moody Lie algebra $\hat{\mathfrak{sl}}_e$ with the vertices numbered clockwise from 1 to e, and encode the fixed ordering on the representatives by a clockwise orientation of the diagram. Recall our $\mathbf{i} \in \mathbb{Z}^n$ from Definition 3.6.

Definition 7.1. We denote by $\mathbf{R_i}$ the *quiver Hecke algebra* associated to $\mathbf{i}$. This is the unital $\Bbbk$-algebra generated by elements

$$\{e(\mathbf{u}) \mid \mathbf{u} \in \mathfrak{S}\mathbf{i}\} \cup \{\psi_1, \ldots, \psi_{n-1}\} \cup \{x_1, \ldots, x_n\}$$

subject to relations

$$e(\mathbf{u})e(\mathbf{u}') = \delta_{\mathbf{u},\mathbf{u}'}e(\mathbf{u}); \qquad \sum_{\mathbf{u}\in\mathfrak{S}\mathbf{i}} e(\mathbf{u}) = 1;$$

$$x_r e(\mathbf{u}) = e(\mathbf{u})x_r; \qquad \psi_r e(\mathbf{u}) = e(s_r \cdot \mathbf{u})\psi_r; \quad x_r x_s = x_s x_r;$$

$$\psi_r\psi_s = \psi_s\psi_r \qquad \text{if } |r-s|>1;$$

$$\psi_r x_s = x_s\psi_r \qquad \text{if } s \neq r, r+1;$$

$$\psi_r x_{r+1}e(\mathbf{u}) = (x_r\psi_r + \delta_{u_r,u_{r+1}})e(\mathbf{u}); \quad x_{r+1}\psi_r e(\mathbf{u}) = (\psi_r x_r + \delta_{u_r,u_{r+1}})e(\mathbf{u});$$

$$\psi_r^2 e(\mathbf{u}) = \begin{cases} 0 & \text{if } u_r = u_{r+1}, \\ e(\mathbf{u}) & \text{if } u_{r+1} \neq u_r \pm 1, u_r, \\ (x_{r+1}-x_r)e(\mathbf{u}) & \text{if } u_{r+1} = u_r + 1, e \neq 2, \\ (x_r - x_{r+1})e(\mathbf{u}) & \text{if } u_{r+1} = u_r - 1, e \neq 2; \\ (x_{r+1}-x_r)(x_r-x_{r+1})e(\mathbf{u}) & \text{if } u_{r+1} = -u_r, e = 2 \end{cases}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{u}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1}+1)e(\mathbf{u}) & \text{if } u_{r+2} = u_r = u_{r+1}-1, e \neq 2, \\ (\psi_{r+1}\psi_r\psi_{r+1}-1)e(\mathbf{u}) & \text{if } u_{r+2} = u_r = u_{r+1}+1, e \neq 2, \\ (\dagger) & \text{if } u_{r+2} = u_r = -u_{r+1}, e = 2, \\ \psi_{r+1}\psi_r\psi_{r+1}e(\mathbf{u}) & \text{otherwise.} \end{cases}$$

where $(\dagger) = (\psi_{r+1}\psi_r\psi_{r+1} - x_r - x_{r+2} + 2x_{r+1})e(\mathbf{u})$.

The commutative subalgebra of $\mathbf{R_i}$ generated by $\{e(\mathbf{u}) \mid \mathbf{u} \in \mathfrak{S}\mathbf{i}\} \cup \{x_1, \dots, x_n\}$ is denoted by $\mathbf{P_i}$.

The following can be found in [KL09] or [Rou08] and can be easily verified.

Lemma 7.2. *The algebra $\mathbf{R_i}$ has a faithful representation on*

$$\mathbf{F_i} = \bigoplus_{\mathbf{u}\in\mathfrak{S}\mathbf{i}} e(\mathbf{u})\Bbbk[x_1, \dots x_n] \cdot \mathbb{1}$$

where the action of $\mathbf{P_i}$ is the regular action and

$$\psi_r e(\mathbf{u}) \cdot \mathbb{1} \quad = \quad \begin{cases} 0 & \textit{if } u_r = u_{r+1}, \\ (x_r - x_{r+1})e(s_k \cdot \mathbf{u}) \cdot \mathbb{1} & \textit{if } u_{r+1} = u_r + 1, \\ e(s_k \cdot \mathbf{u}) \cdot \mathbb{1} & \textit{if } u_{r+1} \neq u_r, u_r + 1. \end{cases} \tag{7.1}$$

Again, we complete our algebra, this time at the sequence of ideals $\mathbf{J}_m = \mathbf{R_i}\mathbf{I}^m\mathbf{R_i}$ where $\mathbf{I}$ is the ideal in $\Bbbk[x_1, \dots x_n]$ generated by all $x_i, i = 1 \dots n$. We denote the completed algebra by $\widehat{\mathbf{R}}_\mathbf{i}$, its polynomial subalgebra generated by $\{e(\mathbf{u})|\mathbf{u} \in \mathfrak{S}\mathbf{i}\} \cup \{x_1, \dots, x_n\}$ by $\widehat{\mathbf{P}}_\mathbf{i}$ and complete our faithful representation to obtain $\widehat{\mathbf{F}}_\mathbf{i} = \widehat{\mathbf{R}}_\mathbf{i} \otimes_{\mathbf{R_i}} \mathbf{F_i}$.

7.2. **The isomorphism $\widehat{\mathcal{H}}_\mathbf{i} \cong \widehat{\mathbf{R}}_\mathbf{i}$.** In this section, we provide an explicit isomorphism between the completed algebras $\widehat{\mathcal{H}}_\mathbf{i}$ and $\widehat{\mathbf{R}}_\mathbf{i}$. A similar result can be found in [Web13]. Note that our approach differs from that used in [Web13],

in that we do not use exponentials, but rather an affine shift following the ideas of [BK09].

First observe that there is an isomorphism

$$\gamma : \widehat{\mathcal{P}}_{\mathbf{i}} \to \widehat{\mathbf{P}}_{\mathbf{i}} : \qquad (X_i - q^{u_i})e_{\mathbf{u}} \mapsto -q^{u_i} x_i e(\mathbf{u}) \tag{7.2}$$

which induces an isomorphism

$$\widehat{\overline{\mathbb{P}}(\mathcal{H})}_{\mathbf{i}} \to \widehat{\mathbf{F}}_{\mathbf{i}} : \qquad \prod_{i=1}^{n} X_i^{a_i} e_{\mathbf{u}} \overline{\mathbf{v}} \mapsto \prod_{i=1}^{n} (q^{u_i}(1-x_i))^{a_i} e(\mathbf{u})$$

between the restrictions of the respective faithful representations to the subalgebras $\widehat{\mathcal{P}}_{\mathbf{i}}$ respectively $\widehat{\mathbf{P}}_{\mathbf{i}}$.

Theorem 7.3. *The isomorphism γ from (7.2) can be extended to an isomorphism of algebras $\gamma : \widehat{\mathcal{H}}_{\mathbf{i}} \to \widehat{\mathbf{R}}_{\mathbf{i}}$, via $\gamma(e_{s_r \cdot \mathbf{u}} \Phi_r) = A_r^{\mathbf{u}} \psi_r e(\mathbf{u})$ where*

$$A_r^{\mathbf{u}} = \begin{cases} 1 - q - x_r + q x_{r+1} & \text{if } u_{r+1} = u_r, \\ \frac{-q}{(1-q-x_{r+1}+qx_r)} & \text{if } u_{r+1} = u_r + 1, \\ \frac{q^{u_{r+1}}(1-x_r) - q^{u_r+1}(1-x_{r+1})}{q^{u_r}(1-x_{r+1}) - q^{u_{r+1}}(1-x_r)} & \text{if } u_{r+1} \neq u_r, u_r + 1 \end{cases}$$

Proof. Since symmetric formal power series in the X_i respectively the x_i commute with the Φ_r from (3.10) respectively the generators ψ_r, it suffices to check the following two types of equalities

$$\gamma(s_r \cdot e_{\mathbf{u}} \Phi_r \overline{\mathbf{v}}) = A_r^{\mathbf{u}} \psi_r e(\mathbf{u}) \cdot \mathbb{1}, \tag{7.3}$$
$$\gamma(e_{s_r \cdot \mathbf{u}} \Phi_r (X_{r+1} - X_r) \overline{\mathbf{v}}) = A_r^{\mathbf{u}} \psi_r \gamma((X_{r+1} - X_r) e_{\mathbf{u}}) \cdot \mathbb{1}, \tag{7.4}$$

for any possible choices of r and $\mathbf{u}$. We start with the equalities (7.3).

i.) If $u_{r+1} = u_r$, then the right hand side of (7.3) is zero by (7.1) whereas the left hand side is zero by (3.14).

ii.) In case $u_{r+1} \neq u_r$, we obtain from (3.16) and the definition in (7.2) that

$$\begin{aligned} \gamma\left(e_{s_r \cdot \mathbf{u}} \Phi_r \overline{\mathbf{v}}\right) &= \gamma\left(\frac{X_r - q X_{r+1}}{X_{r+1} - X_r} e_{s_r \cdot \mathbf{u}} \overline{\mathbf{v}}\right) \\ &= \frac{q^{u_{r+1}}(1-x_r) - q^{u_r+1}(1-x_{r+1})}{q^{u_r}(1-x_{r+1}) - q^{u_{r+1}}(1-x_r)}\, e(s_r \cdot \mathbf{u}) \cdot \mathbb{1}. \end{aligned} \tag{7.5}$$

a.) If $u_{r+1} = u_r + 1$, we can simplify (7.5) to

$$\begin{aligned} \gamma(e_{s_r \cdot \mathbf{u}} \Phi_r \overline{\mathbf{v}}) &= \frac{q^{u_r+1}(1-x_r) - q^{u_r+1}(1-x_{r+1})}{q^{u_r}(1-x_{r+1}) - q^{u_r+1}(1-x_r)} e(s_r \cdot \mathbf{u}) \cdot \mathbb{1} \\ &= \frac{q^{u_r+1}(x_{r+1} - x_r)}{q^{u_r}(1 - q - x_{r+1} + q x_r)} e(s_r \cdot \mathbf{u}) \cdot \mathbb{1} \\ &= \frac{q(x_{r+1} - x_r)}{(1 - q - x_{r+1} + q x_r)} e(s_r \cdot \mathbf{u}) \cdot \mathbb{1} \end{aligned}$$

On the other hand, in this case $A_r^{\mathbf{u}}\psi_r e(\mathbf{u})\cdot\mathbb{1} = A_r^{\mathbf{u}}(x_r - x_{r+1})e(s_r\cdot\mathbf{u})\cdot\mathbb{1}$, so our choice of $A_r^{\mathbf{u}} = \frac{-q}{(1-q-x_{r+1}+qx_r)}$ again means that (7.3) holds.

b.) If $u_{r+1}\neq u_r, u_r+1$ then $A_r^{\mathbf{u}}\psi_r e(\mathbf{u})\cdot\mathbb{1} = A_r^{\mathbf{u}}e(s_r\cdot\mathbf{u})\cdot\mathbb{1}$, we see that (7.3) indeed holds if we set $A_r^{\mathbf{u}} = \frac{q^{u_r+1}(1-x_r)-q^{u_r+1}(1-x_{r+1})}{q^{u_r}(1-x_{r+1})-q^{u_{r+1}}(1-x_r)}$.

This settles (7.3), and we proceed to (7.4). The left hand side equals

$$\gamma((X_{r+1}-X_r)e_{\mathbf{u}}) \quad = \quad (q^{u_{r+1}}(1-x_{r+1}) - q^{u_r}(1-x_r))e(\mathbf{u}).$$

If $u_{r+1}\neq u_r$ then, using (3.15), we obtain the equalities

$$\begin{aligned} LHS &= \gamma(e_{s_r\cdot\mathbf{u}}\Phi_r(X_{r+1}-X_r)\overline{\mathbf{v}}) \\ &= \gamma(qX_{r+1}-X_r)e_{s_r\cdot\mathbf{u}}\overline{\mathbf{v}}) \\ &= (q^{u_r+1}(1-x_{r+1}) - q^{u_{r+1}}(1-x_r))e(s_r\mathbf{u})\cdot\mathbb{1} \end{aligned} \tag{7.6}$$

On the other hand, in this case, the right hand side equals

$$\begin{aligned} RHS &= A_r^{\mathbf{u}}\psi_r\gamma((X_{r+1}-X_r)e_{\mathbf{u}})\cdot\mathbb{1} \\ &= A_r^{\mathbf{u}}\psi_r(q^{u_{r+1}}(1-x_{r+1}) - q^{u_r}(1-x_r))e(\mathbf{u})\cdot\mathbb{1} \\ &= A_r^{\mathbf{u}}(q^{u_{r+1}}(1-x_r) - q^{u_r}(1-x_{r+1}))\psi_r e(\mathbf{u})\cdot\mathbb{1} \\ &= A_r^{\mathbf{u}}(q^{u_{r+1}}(1-x_r) - q^{u_r}(1-x_{r+1}))e(s_r\cdot\mathbf{u})\cdot\mathbb{1} \end{aligned} \tag{7.7}$$

i.) If $u_{r+1}\neq u_r, u_r+1$, then $A_r^{\mathbf{u}} = \frac{q^{u_r+1}(1-x_{r+1})-q^{u_{r+1}}(1-x_r)}{q^{u_{r+1}}(1-x_r)-q^{u_r}(1-x_{r+1})}$ gives (7.4).

ii.) If $u_{r+1} = u_r+1$, then the desired equation simplifies via (7.1) to

$$\begin{aligned} &(q^{u_r+1}(x_r - x_{r+1}r))e(s_r\cdot\mathbf{u})\cdot\mathbb{1} \\ &\qquad = A_r^{\mathbf{u}}(q^{u_r}(q-1-qx_r+x_{r+1}))(x_r-x_{r+1})e(s_r\cdot\mathbf{u})\cdot\mathbb{1}. \end{aligned}$$

Hence (7.4) holds if we set $A_r^{\mathbf{u}} = \frac{q}{q-1-qx_r+x_{r+1}}$.

iii.) If $u_{r+1} = u_r$, then, using (3.15) we have

$$\begin{aligned} LHS &= \gamma(e_{s_r\cdot\mathbf{u}}\Phi(X_{r+1}-X_r)\overline{\mathbf{v}}) \\ &= \gamma(2(qX_{r+1}-X_r))e(\mathbf{u})\cdot\mathbb{1}) \\ &= 2(q^{u_r+1}(1-x_{r+1}) - q^{u_r}(1-x_r))e(\mathbf{u})\cdot\mathbb{1}) \\ &= 2q^{u_r}(q(1-x_{r+1}) - (1-x_r))e(\mathbf{u})\cdot\mathbb{1}) \end{aligned}$$

for the left hand side of our formula (7.4), while the right hand side becomes

$$\begin{aligned} RHS &= A_r^{\mathbf{u}}\psi_r(q^{u_{r+1}}(1-x_{r+1}) - q^{u_r}(1-x_r))e(\mathbf{u})\cdot\mathbb{1} \\ &= A_r^{\mathbf{u}}\psi_r(q^{u_r}(x_r - x_{r+1}r)e(\mathbf{u})\cdot\mathbb{1} = -2q^{u_r}A_r^{\mathbf{u}}e(\mathbf{u})\cdot\mathbb{1}. \end{aligned}$$

Therefore, setting $A_r^{\mathbf{u}} = (1-x_r) - q(1-x_{r+1}) = 1-q-x_r+qx_{r+1}$ implies that (7.4) also holds in this final case.

Hence the theorem is proved. $\square$

8. Quiver Schur algebras

In this section we establish our main isomorphism theorem by connecting the affine Schur algebra (defined by Vignéras) with the quiver Schur algebra from [SW11]. We do this via an auxiliary *modified* quiver Schur algebra which we now introduce.

8.1. **The modified quiver Schur algebra.** Recall $\mathbf{i}$ from Definition 3.6. For $J \in \mathbb{I}$, and

$$\mathbf{u}_J = (u_1, \cdots, u_{t_1} | u_{t_1+1}, \cdots, u_{t_2} | \cdots | u_{t_{r-1}+1}, \cdots, u_{t_r}) \in \mathbf{U}_J$$

with dimension vector $\mathtt{d} = \mathtt{d}(\mathbf{i}) = \mathtt{d}(\mathbf{u}, J) = (d_1, d_2, \ldots, d_e)$ we define

$$\Lambda_{\mathbf{u}_J} = \Bbbk[y_{1,1}, \ldots, y_{1,d_1}, y_{2,1}, \ldots, y_{2,d_2}, \ldots, y_{e,d_e}]^{\mathfrak{S}_{\mathbf{u}_J}}$$

and set

$$\Lambda := \bigoplus_{\substack{J \subseteq \mathbb{I} \\ \mathbf{u}_J \in \mathbf{U}_J}} \Lambda_{\mathbf{u}_J}.$$

Definition 8.1. Let $1 \leq i \leq e$. For $1 \leq k \leq r$ let $c(k)_i = \sum_{j=1}^{k} d_i^j$, using Definition 5.7 and $c(0)_i = 0$. Then the *total reversed Euler class attached to* $\mathbf{u}_J$ is

$$\mathtt{E}_{\mathbf{u}_J} \quad = \quad \prod_{i=1}^{e} \prod_{s=1}^{r-1} \prod_{j=c(s-1)_i+1}^{c(s)_i} \prod_{k=c(s)_{i+1}+1}^{d_{i+1}} (y_{i,j} - y_{i+1,k}). \tag{8.1}$$

with $\mathtt{E}_{\mathbf{u}_{\mathbb{I}}} := 1$. The *total symmetriser* is defined as as

$$\mathtt{S}_{\mathbf{u}_J} \quad = \quad \prod_{i=1}^{e} \prod_{s=1}^{r-1} \prod_{j=c(s-1)_i+1}^{c(s)_i} \prod_{k=c(s)_i+1}^{d_i} (y_{i,j} - y_{i,k}) \tag{8.2}$$

with $\mathtt{S}_{\mathbf{u}_{\mathbb{I}}} := 1$. More generally, assume $J \subset K$ and let $\mathbf{u}_K \in \mathbf{U}_K$ be a merge of $\mathbf{u}_J$. Then their *relative reversed Euler class* respectively the *relative symmetriser* are defined as

$$\mathtt{E}_{\mathbf{u}_J}^{\mathbf{u}_K} \;=\; \frac{\mathtt{E}_{\mathbf{u}_J}}{\mathtt{E}_{\mathbf{u}_K}} \qquad \text{respectively} \qquad \mathtt{S}_{\mathbf{u}_J}^{\mathbf{u}_K} \;=\; \frac{\mathtt{S}_{\mathbf{u}_J}}{\mathtt{S}_{\mathbf{u}_K}}. \tag{8.3}$$

In particular, the special case $K = \mathbb{I}$ gives the total reversed Euler class respectively the total symmetriser. Note that the relative Euler class and relative symmetriser are again polynomials.

Example 8.2. Note that $\mathtt{E}_{(1|2)} = y_{1,1} - y_{2,1}$, and $\mathtt{E}_{(2|1)} = 1$ if $e > 2$ whereas $\mathtt{E}_{(2|1)} = y_{2,1} - y_{1,1}$ if $e = 2$ and $\mathtt{E}_{(1|1)} = 1$. Moreover, $\mathtt{S}_{(1|2)} = \mathtt{S}_{(2|1)} = 1$, whereas $\mathtt{E}_{(1|1)} = y_{1,1} - y_{1,2}$.

Example 8.3. Let for instance $\mathbf{u}_J = (1,2|1,1|1,2|1)$, $\mathbf{u}_K = (1,2|1,1,1,2|1)$. Then for $e \geq 3$ we have $\mathtt{E}_{\mathbf{u}_J} = (y_{1,1} - y_{2,2})(y_{1,2} - y_{2,2})(y_{1,3} - y_{2,2}) =: E$ and $\mathtt{E}_{\mathbf{u}_K} = (y_{1,1} - y_{2,2})$ and therefore $\mathtt{E}_{\mathbf{u}_J}^{\mathbf{u}_K} = (y_{1,2} - y_{2,2})(y_{1,3} - y_{2,2})$, whereas for $e = 2$ we have $\mathtt{E}_{\mathbf{u}_J} = E(y_{2,1} - y_{1,2})(y_{2,1} - y_{1,3})(y_{2,1} - y_{1,4})(y_{2,1} - y_{1,5})(y_{2,2} - y_{1,5})$

but again $\mathtt{E}_{\mathbf{u}_J}^{\mathbf{u}_K} = (y_{1,2} - y_{2,2})(y_{1,3} - y_{2,2})$. On the other hand, for any $e \geq 2$, we have $\mathtt{S}_{\mathbf{u}_J}^{\mathbf{u}_K} = (y_{1,2} - y_{1,4})(y_{1,3} - y_{1,4})$.

Definition 8.4. We define the *modified quiver Schur algebra* $\mathbf{C}_{\mathbf{i}}$ as the subalgebra of $\mathrm{End}_{\Bbbk}(\Lambda)$ generated by the following endomorphisms:

- the *idempotents*: $e(\mathbf{u}_J)$ for $\mathbf{u}_J \in \mathbf{U}_J$ for any J, defined as the projection onto the summand $\Lambda_{\mathbf{u}_J}$,
- the *polynomial*: $e(\mathbf{u}_J)pe(\mathbf{u}_J)$ for $\mathbf{u}_J \in \mathbf{U}_J$ for any J, and $p \in \Lambda_{\mathbf{u}_J}$, defined as multiplication by p on the summand $\Lambda_{\mathbf{u}_J}$.
- the *splits*: $\curlyvee_{\mathbf{u}_K}^{\mathbf{u}_J}$ for $J \subset K$, $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, thus defining $\mathbf{u}_K \in \mathbf{U}_K$ sending any element $f \in \Lambda_{\mathbf{u}_K}$ to $f \in \Lambda_{\mathbf{u}_J}$ and any $f \in \Lambda_{\mathbf{u}'_{K'}}$ with $\mathbf{u}'_{K'} \neq \mathbf{u}_K$ to zero. In other words, a split is just the embedding of the summand $\Lambda_{\mathbf{u}_K}$ into the summand $\Lambda_{\mathbf{u}_J}$.
- the *merges*: $\curlywedge_{\mathbf{u}_J}^{\mathbf{u}_K}$ for any $J \subset K$ $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, thus defining $\mathbf{u}_K \in \mathbf{U}_K$, defined on an element $f \in \Lambda_{\mathbf{u}'_{J'}}$ by
$$f \quad \mapsto \begin{cases} \Delta(\mathtt{E}_{\mathbf{u}_J}^{\mathbf{u}_K} f) \in \Lambda_{\mathbf{u}_K} & \text{if } \mathbf{u}_J = \mathbf{u}'_{J'}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.4}$$
where $\Delta = \Delta_{\mathbf{u}_J}^{\mathbf{u}_K}$ sends an element f to the total invariant $\pitchfork_{I_{\mathbf{u}_J}}^{I_{\mathbf{u}_K}} \left(\frac{f}{\mathtt{S}_{\mathbf{u}_J}^{\mathbf{u}_K}} \right)$.

Using a reformulation in terms of Demazure operators, see Proposition 8.13, it follows that $\pitchfork_{I_{\mathbf{u}_J}}^{I_{\mathbf{u}_K}} \left(\frac{f}{\mathtt{S}_{\mathbf{u}_J}^{\mathbf{u}_K}} \right)$ is indeed again a polynomial.

Example 8.5. Consider for instance $\mathbf{u}_J = (1|1)$ and $\mathbf{u}_K = (11)$. Then for $f \in \Lambda_{\mathbf{u}_K} = \Bbbk[y_{1,1}, y_{1,2}]$ we have $\curlywedge_{\mathbf{u}_J}^{\mathbf{u}_K}(f) = \Delta(\frac{f}{y_{1,1} - y_{1,2}}) = 2f_2$, where $f = f_1 + (y_{1,1} - y_{1,2})f_2$ with (uniquely determined) $f_1, f_2 \in \Lambda_{\mathbf{u}_J}$.

Example 8.6. Let us describe the merge endomorphism explicitly in the simplest case where $\mathbf{u}_J = (1^{a_1}, 2^{a_2}, \ldots, e^{a_e} | 1^{b_1}, 2^{b_2}, \ldots, e^{b_e})$ has only two parts, hence $\mathbf{u}_K = (1^{d_1}, 2^{d_2}, \ldots, e^{d_e})$ with $d_i = a_i + b_i$. Then our formulae give
$$\Delta(\mathtt{E}_{\mathbf{u}_J}^{\mathbf{u}_K} f) \quad = \quad \pitchfork_{a_1, b_1, a_2, b_2, \ldots, a_e, b_e}^{d_1, \quad d_2, \quad \ldots, d_e} \left(\frac{\prod_{i=1}^{e} \prod_{j=1}^{a_i} \prod_{k=a_{i+1}+1}^{d_{i+1}} (y_{i,j} - y_{i+1,k})}{\prod_{i=1}^{e} \prod_{j=1}^{a_i} \prod_{k=a_i+1}^{d_i} (y_{i,j} - y_{i,k})} f \right).$$

We denote by $\widehat{\mathbf{C}}_{\mathbf{i}}$ the completion of $\mathbf{C}_{\mathbf{i}}$ at the ideal generated by all $e(\mathbf{u}_J)pe(\mathbf{u}_J)$ for all $J \subset \mathbb{I}, \mathbf{u}_J \in \mathbf{U}_J$ and all $p \in \Lambda_{\mathbf{u}_J}$ with zero constant term. Then $\widehat{\mathbf{C}}_{\mathbf{i}}$ has a faithful representation on
$$\widehat{\Lambda} = \bigoplus_{\substack{J \subseteq \mathbb{I} \\ \mathbf{u}_J \in \mathbf{U}_J}} \widehat{\Lambda}_{\mathbf{u}_J}$$

where $\widehat{\Lambda}_{\mathbf{u}_J}$ denotes the completion of $\Lambda_{\mathbf{u}_J}$ at the ideal of all polynomials with zero constant term.

8.2. **The quiver Schur algebra.** Here we recall the definition of the quiver Schur algebra $\mathbf{A_i}$, introduced by the second author and Webster in [SW11]. For $J \in \mathbb{I}$, and

$$\mathbf{u}_J = (u_1, \cdots, u_{t_1} | u_{t_1+1}, \cdots, u_{t_2} | \cdots | u_{t_{r-1}+1}, \cdots, u_{t_r}) \in \mathbf{U}_J$$

with dimension vector $\mathtt{d} = \mathtt{d}(\mathbf{i}) = \mathtt{d}(\mathbf{u}, J) = (d_1, d_2, \ldots, d_e)$ we define

$$\mathring{\Lambda}_{\mathbf{u}_J} \quad = \quad \Bbbk[z_{1,1}, \ldots, z_{1,\gamma_1}, z_{2,1}, \ldots, z_{2,\gamma_2}, \ldots z_{e,\gamma_e}]^{\mathfrak{S}_{\mathbf{u}_J}} \tag{8.5}$$

and set

$$\mathring{\Lambda} := \bigoplus_{\substack{J \subseteq \mathbb{I} \\ \mathbf{u}_J \in \mathbf{U}_J}} \mathring{\Lambda}_{\mathbf{u}_J}.$$

Definition 8.7. Let $1 \leq i \leq e$. For $1 \leq k \leq r$ let $c(k)_i = \sum_{j=1}^{k} d_i^j$, using Definition 5.7, and $c(0)_i = 0$. Then the *total Euler class* and *the symmetriser* for $\mathbf{u}_J$ are defined as

$$\mathring{\mathtt{E}}_{\mathbf{u}_J} \quad = \quad \prod_{i=1}^{e} \prod_{s=1}^{r-1} \prod_{j=c(s-1)_{i+1}+1}^{c(s)_{i+1}} \prod_{k=c(s)_i+1}^{d_i} (z_{i+1,j} - z_{i,k}).$$

(notice that this is $\mathfrak{S}_{\mathbf{u}_J}$-invariant), respectively

$$\mathring{\mathtt{S}}_{\mathbf{u}_J} \quad = \quad \prod_{i=1}^{e} \prod_{s=1}^{r-1} \prod_{j=c(s-1)_i+1}^{c(s)_i} \prod_{k=c(s)_i+1}^{d_i} (z_{i,j} - z_{i,k})$$

(Note that $\mathring{\mathtt{S}}_{\mathbf{u}_J}$ is the same as $\mathtt{S}_{\mathbf{u}_J}$, only written in variables $z_{i,j}$ instead of $y_{i,j}$.)

More generally, assume $J \subset K$ and let $\mathbf{u}_K \in \mathbf{U}_K$ be a merge of $\mathbf{u}_J$. Then their *relative Euler class* respectively the *relative symmetriser* are defined as

$$\mathring{\mathtt{E}}_{\mathbf{u}_J}^{\mathbf{u}_K} \ = \ \frac{\mathring{\mathtt{E}}_{\mathbf{u}_J}}{\mathring{\mathtt{E}}_{\mathbf{u}_K}} \qquad \text{respectively} \qquad \mathring{\mathtt{S}}_{\mathbf{u}_J}^{\mathbf{u}_K} \ = \ \frac{\mathring{\mathtt{S}}_{\mathbf{u}_J}}{\mathring{\mathtt{S}}_{\mathbf{u}_K}}.$$

The following was introduced in [SW11].

Definition 8.8. The *quiver Schur algebra* $\mathbf{A_i}$ is the subalgebra of $\mathrm{End}_{\Bbbk}(\mathring{\Lambda})$ generated by the following endomorphisms:

- the *idempotents*: $e(\mathbf{u}_J)$ for $\mathbf{u}_J \in \mathbf{U}_J$ for any J, defined as the projection onto the summand $\mathring{\Lambda}_{\mathbf{u}_J}$,
- the *polynomial*: $e(\mathbf{u}_J)pe(\mathbf{u}_J)$ for any J and $p \in \mathring{\Lambda}_{\mathbf{u}_J}$, defined as multiplication by p on the summand $\mathring{\Lambda}_{\mathbf{u}_J}$.

- the *splits*: $\mathring{\curlyvee}^{\mathbf{u}_J}_{\mathbf{u}_K}$ for any $J \subset K$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, defined on an element f with $f \in \mathring{\Lambda}_{\mathbf{u}'_{K'}}$ by

$$f \quad \mapsto \begin{cases} \mathring{\mathbb{E}}^{\mathbf{u}_K}_{\mathbf{u}_J} f \in \mathring{\Lambda}_{\mathbf{u}_J} & \text{if } \mathbf{u}'_{K'} = \mathbf{u}_K, \\ 0 & \text{otherwise.} \end{cases}$$

  In other words, a split is just the embedding of the summand $\mathring{\Lambda}_{\mathbf{u}_K}$ into the summand $\mathring{\Lambda}_{\mathbf{u}_J}$, followed by multiplication with $\mathring{\mathbb{E}}^{\mathbf{u}_K}_{\mathbf{u}_J}$.
- the *merges*: $\mathring{\curlywedge}^{\mathbf{u}_K}_{\mathbf{u}_J}$ for any $J \subset K$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, defined on an element $f \in \mathring{\Lambda}_{\mathbf{u}'_{J'}}$ by

$$f \quad \mapsto \begin{cases} \mathring{\Delta}(f) \in \mathring{\Lambda}_{\mathbf{u}_K} & \text{if } \mathbf{u}'_{J'} = \mathbf{u}_J, \\ 0 & \text{otherwise.} \end{cases} \tag{8.6}$$

  where $\mathring{\Delta} = \mathring{\Delta}^{\mathbf{u}_K}_{\mathbf{u}_J}$ sends an element f to the total invariant

$$\mathring{\Delta}(f) \quad = \quad \pitchfork^{I_{\mathbf{u}_K}}_{I_{\mathbf{u}_J}} \left(\frac{f}{\mathbb{S}^{\mathbf{u}_K}_{\mathbf{u}_J}} \right). \tag{8.7}$$

Note that again the translation into Demazure operators from Proposition 8.13 ensures that $\mathring{\Delta}(f)$ is in fact a polynomial.

Again we have simple splits and merges: In case $|K \setminus J| = 1$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, we call $\mathring{\curlyvee}^{\mathbf{u}_J}_{\mathbf{u}_K}$ a *simple split* and $\mathring{\curlywedge}^{\mathbf{u}_K}_{\mathbf{u}_J}$ a *simple merge*. If $K = \mathbb{I}$ and $J = K \setminus \{a\}$, we also denote these by $\mathring{\curlyvee}^{\mathbf{a}+\mathbf{b}}_{\mathbf{a},\mathbf{b}}$ respectively $\mathring{\curlywedge}^{\mathbf{a},\mathbf{b}}_{\mathbf{a}+\mathbf{b}}$, where $a_i = d_i^1$ and $b_i = d_i^2$ with the notation from Definition 5.7.

Remark 8.9. In [SW11], the quiver Schur algebra was only defined over the complex numbers, since the involved geometry would require more advanced tools. However one can easily verify that the faithful representation defined in [SW11] makes sense over any field and so we just *define* the quiver Schur algebra over an arbitrary field as in Definition 8.8. In characteristic zero it agrees with the one defined in [SW11] by Remark 8.11 below.

Example 8.10. Note that in the case of a simple merge $\mathring{\curlywedge}^{\mathbf{u}_K}_{\mathbf{u}_J}$ of the form $\mathbf{u}_J = (1^{a_1}, 2^{a_2}, \dots, e^{a_e} | 1^{b_1}, 2^{b_2}, \dots, e^{b_e})$ and $K = \mathbb{I}$, hence $\mathbf{u}_K = (1^{d_1}, 2^{d_2}, \dots, e^{d_e})$, the formula (8.6) simplifies to

$$\begin{aligned} \mathring{\curlywedge}^{\mathbf{u}_K}_{\mathbf{u}_J}(f) \quad &= \quad \mathring{\curlywedge}^{\mathbf{a}+\mathbf{b}}_{\mathbf{a},\mathbf{b}}(f) \;=\; \mathring{\Delta}(f) \\ &= \quad \pitchfork^{d_1,\dots,d_e}_{a_1,b_1,a_2,b_2,\dots,a_e,b_e} \left(\frac{f}{\prod_{i=1}^{e} \prod_{j=1}^{a_i} \prod_{k=a_i+1}^{a_i+d_i} (z_{i,j} - z_{i,k})} \right). \end{aligned}$$

with the relative Euler class

$$\mathring{\mathrm{E}}_{\mathbf{u}'_J}^{\mathbf{u}_K} = \prod_{i=1}^{e} \prod_{k=1}^{c_{i+1}} \prod_{j=c_i+1}^{d_i+c_i} (z_{i+1,k} - z_{i,j}).$$

This, indeed, corresponds to the formulae given in [SW11].

Remark 8.11. Assume $K = \mathbb{I}$ and $J = K \setminus \{a\}$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$. Let $(\mathbf{a}, \mathbf{b})$ be the dimension vector of $\mathbf{u}$, where $a_i = d_i^1$ and $b_i = d_i^2$ with the notation from Definition 5.7. Write $S_{\mathbf{a}+\mathbf{b}}$ for $W_{I_{\mathbf{u}_K}}$ and $S_{\mathbf{a},\mathbf{b}}$ for $W_{I_{\mathbf{u}_J}}$. In [SW11], the action of a simple merge $\mathring{\curlywedge}_{\mathbf{a},\mathbf{b}}^{\mathbf{a}+\mathbf{b}}$ on the faithful representation (8.5) is defined as sending f to

$$\sum_{w \in S_{\mathbf{a}+\mathbf{b}}} (-1)^{l(w)} w(f) \prod_{i=1}^{e} \frac{1}{a_i! b_i!} \frac{w\Big(\prod_{1 \le j < k \le a_i} (z_{i,j} - z_{i,k}) \prod_{a_i < j < k \le a_i + b_i} (z_{i,j} - z_{i,k})\Big)}{\prod_{1 \le j < k \le a_i + b_i} (z_{i,j} - z_{i,k})}.$$

Note that, in contrast to formula (8.6), this expression does not make sense in positive characteristic in general. In characteristic zero however, this expression coincides with (8.6), since we have

$$\begin{aligned}
& \sum_{w \in S_{\mathbf{a}+\mathbf{b}}} w(f) \prod_{i=1}^{e} \frac{1}{a_i! b_i!} w\left(\frac{\prod_{1 \le j < k \le a_i} (z_{i,j} - z_{i,k}) \prod_{a_i < j < k \le a_i + b_i} (z_{i,j} - z_{i,k})}{\prod_{1 \le j < k \le a_i + b_i} (z_{i,j} - z_{i,k})}\right) \\
= & \sum_{w \in S_{\mathbf{a}+\mathbf{b}}} w(f) \prod_{i=1}^{e} \frac{1}{a_i! b_i!} w\left(\frac{1}{\prod_{j=1}^{a_i} \prod_{k=a_i+1}^{a_i+b_i} (z_{i,j} - z_{i,k})}\right) \\
= & \sum_{w \in D_{\emptyset, I_{\mathbf{u}_J}}^{I_{\mathbf{u}_K}}} w(f) \prod_{i=1}^{e} w\left(\frac{1}{\prod_{j=1}^{a_i} \prod_{k=a_i+1}^{a_i+b_i} (z_{i,j} - z_{i,k})}\right) \\
= & \sum_{w \in D_{\emptyset, I_{\mathbf{u}_J}}^{I_{\mathbf{u}_K}}} w\left(f \prod_{i=1}^{e} \frac{1}{\prod_{j=1}^{a_i} \prod_{k=a_i+1}^{a_i+b_i} (z_{i,j} - z_{i,k})}\right) \\
= & \pitchfork_{a_1,\ b_1, a_2,\ b_2, \dots, a_e,\ b_e}^{a_1+b_1, a_2+b_2, \dots, a_e+b_e} \left(f \prod_{i=1}^{e} \frac{1}{\prod_{j=1}^{a_i} \prod_{k=a_i+1}^{a_i+b_i} (z_{i,j} - z_{i,k})}\right)
\end{aligned}$$

as desired.

8.3. **Demazure or Bernstein-Gelfand-Gelfand difference operators.** In this subsection we connect our merging formulae to the classical difference operators. For $1 \le i \le n-1$, the ith *Demazure operator* or *divided difference operator* from [Dem74] or [BGG73] is the endomorphism

$$\Delta_i : \quad \mathbb{k}[X_1, X_2, \dots, X_{n-1}] \to \mathbb{k}[X_1, X_2, \dots, X_{n-1}]$$

defined as

$$\Delta_i(f) = \frac{f - s_i(f)}{X_i - X_{i+1}}.$$

For $f, g \in \Bbbk[X_1, X_2, \dots, X_{n-1}]$, we have $\Delta_i(fg) = \Delta_i(f)g + s_i(f)\Delta_i(g)$, in particular $\Delta_i(fg) = f\Delta_i(g)$ if f is s_i-invariant, and $\Delta_i^2 = 0$. Moreover, for a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, the operator $\Delta_w = \Delta_{s_{i_1}}\Delta_{s_{i_2}}\cdots\Delta_{s_{i_r}}$ is independent of the chosen reduced expression.

Lemma 8.12. *Let $w_0 = w_0(n) \in \mathfrak{S}$ be the longest element, then*

$$\Delta_{w_0}(X_1^{n-1}X_2^{n-2}\cdots X_{n-2}^2 X_{n-1}) \quad = \quad 1.$$

In particular if w_K is the longest element of some parabolic subgroup W_K in $\mathfrak{S}$, then there exists some polynomial h such that $\Delta_{w_K}(h) = 1$.

Proof. If $n = 1$ then $\Delta_1(X_1) = \frac{X_1 - X_2}{X_1 - X_2} = 1$ and so the claim holds. Assuming the formula holds for n, we deduce it for $n{+}1$ by writing first $w_0(n{+}1) = dw_0(n)$ with $d = s_1s_2\cdots s_n$ via Lemma 6.1, and then compute

$$\begin{aligned} & \Delta_{w_0(n+1)}(X_1^{n-1}X_2^{n-2}\cdots X_{n-1}) \;=\; \Delta_d(\Delta_{w_0(n)}(X_1^{n-1}X_2^{n-2}\cdots X_{n-1})) \\ = \;& \Delta_d(\Delta_{w_0(n)}(X_1^{n-2}X_2^{n-3}\cdots X_{n-2})X_1\cdots X_{n-1})) \;=\; \Delta_d(X_1\cdots X_{n-1})) \\ = \;& \Delta_{s_1s_2\cdots s_{n-1}}(X_1X_2\cdots X_{n-2}\Delta_{n-1}(X_{n-1})) \\ = \;& \Delta_{s_1s_2\cdots s_{n-1}}(X_1X_2\cdots X_{n-2}) \;=\; \cdots \;=\; \Delta_{s_1}(X_1) = 1. \end{aligned}$$

Hence the lemma follows. $\square$

There exists in fact a closed formula for Δ_{w_0}, namely

$$\Delta_{w_0} \quad = \quad \frac{1}{\blacktriangle}\sum_{w\in\mathfrak{S}}(-1)^{l(w)}w = \sum_{w\in\mathfrak{S}} w\frac{1}{\blacktriangle}, \tag{8.8}$$

where $\blacktriangle = \prod_{1\leq i<j\leq n}(X_i - X_j)$. The first equality can, for example, be found in [Ful97, Lemma 12], the second equality is an elementary calculation observing that a simple transposition changes the sign of $\blacktriangle$ by -1.

The merges from (8.7) can then be rephrased in terms of Demazure operators as follows (explaining the notations Δ and $\mathring{\Delta}$)

Proposition 8.13. *Assume we are in the setup from* (8.7) *and abbreviate $J' = I_{\mathbf{u}_J}$ and $K' = I_{\mathbf{u}_K}$ using Definition 5.12. Then we have the equality*

$$\pitchfork_J^K\left(\frac{f}{\mathrm{S}_{\mathbf{u}_J}^{\mathbf{u}_K}}\right) \quad = \quad \Delta_{d_J^K}. \tag{8.9}$$

on $\Lambda_{\mathbf{u}_J}$, where $d_J^K \in D_{J'}^{K'}$ is of maximal length (i.e. the representative of the longest element in $W_{J'} \subset W_{K'}$).

Proof. Let $f \in \Lambda_{\mathbf{u}_J}$ and let w_J be the longest element in $W_{J'} \subset W_{K'}$ and w_K the longest element in $W_{K'}$. Then

$$\Delta_{d_J^K}(f) = \Delta_{d_J^K}(f\cdot 1) = \Delta_{d_J^K}(f\cdot\Delta_{w_J}(h)) = \Delta_{d_J^K}(\Delta_{w_J}(fh)) = \Delta_{w_0}(fh)$$

with h as in Lemma 8.12. Here, for the penultimate equality, we have used that f is W_J-invariant and for the last equality that $dw_J = w_K$. With the explicit formula from (8.8), we obtain

$$\begin{aligned}\Delta_{w_0}(fh) &= \frac{1}{\blacktriangle}\sum_{w\in W_{K'}}(-1)^{l(w)}w(fh)\\ &= \frac{1}{\blacktriangle}\left(\sum_{d\in D_{J'}^{K'}}(-1)^{l(d)}d(f)\right)\left(\sum_{w\in W_{J'}}(-1)^{l(w)}w(h)\right),\end{aligned}$$

where the denominator is the Vandermonde determinant

$$\begin{aligned}\blacktriangle &= \prod_{i=1}^{e}\prod_{1\le j<k\le d_i}(z_{i,j}-z_{i,k})\\ &= \left(\prod_{i=1}^{e}\prod_{s=1}^{r-1}\prod_{j=c(s-1)_i+1}^{c(s)_i}\prod_{k=c(s)_i+1}^{d_i}(z_{i,j}-z_{i,k})\right)\\ &\qquad\cdot\left(\prod_{i=1}^{e}\prod_{s=1}^{r-1}\prod_{c(s-1)_i+1\le j<k\le c(s)_i}(z_{i,j}-z_{i,k})\right)\end{aligned}$$

Hence, using (8.8), we obtain $\Delta_{w_0}(fh) = \pitchfork_J^K\left(\frac{f}{\mathbf{S}_{\mathbf{u}_J}^{\mathbf{u}_K}}\right)\Delta_{w_J}(h)$. Now the proposition follows with the definition of h. $\square$

8.4. **The shifted quiver Schur algebra $\mathbf{B_i}$.** We now define the *shifted quiver Schur algebra* $\mathbf{B_i}$ in almost the same way, except that the Euler class moves from the split to the merge. More precisely, define $\mathbf{B_i}$ as the subalgebra of $\mathrm{End}_{\Bbbk}(\mathring{\Lambda})$ generated by the following endomorphisms:

- the *idempotents*: $e(\mathbf{u}_J)$ for $\mathbf{u}_J \in \mathbf{U}_J$ for any J, defined as the projection onto the summand $\mathring{\Lambda}_{\mathbf{u}_J}$,
- the *polynomial*: $e(\mathbf{u}_J)pe(\mathbf{u}_J)$ for any J and $p \in \mathring{\Lambda}_{\mathbf{u}_J}$, defined as multiplication by p on the summand $\mathring{\Lambda}_{\mathbf{u}_J}$.
- the *splits*: $\overset{\rightarrow}{\curlyvee}{}_{\mathbf{u}_K}^{\mathbf{u}'_J}$ for any $J\subset K$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, defined on an element f with $f \in \mathring{\Lambda}_{\mathbf{u}'_{K'}}$ by

$$f \quad\mapsto \begin{cases} f\in\mathring{\Lambda}_{\mathbf{u}_J} & \text{if } \mathbf{u}'_{K'} = \mathbf{u}_K,\\ 0 & \text{otherwise.}\end{cases}$$

  In other words, a split is just the embedding of the summand $\mathring{\Lambda}_{\mathbf{u}_K}$ into the summand $\mathring{\Lambda}_{\mathbf{u}_J}$.

- the *merges*: $\vec{\curlywedge}_{\mathbf{u}_J}^{\mathbf{u}_K}$ for any $J \subset K$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, defined on an element $f \in \mathring{\Lambda}_{\mathbf{u}'_{J'}}$ by

$$f \quad \mapsto \begin{cases} \mathring{\Delta}(\mathring{\mathbb{E}}_{\mathbf{u}_J}^{\mathbf{u}_K} f) \in \mathring{\Lambda}_{\mathbf{u}_K} & \text{if } \mathbf{u}'_{J'} = \mathbf{u}_J, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this algebra is again defined for any field.

9. The main result: The Isomorphism Theorem

The goal of this section is to prove the main Isomorphism Theorem 9.7 between the completed affine Schur algebra and the quiver Schur algebra using the auxiliary algebras in between.

9.1. The isomorphism $\widehat{\mathcal{S}}_{\mathbf{i}} \cong \widehat{\mathbf{C}}_{\mathbf{i}}$.

We now compare the faithful representation of the modified quiver Schur algebra $\mathbf{C}_{\mathbf{i}}$ with the faithful representation of the completed affine Schur algebra $\widehat{\mathcal{S}}_{\mathbf{i}}$.

The following isomorphism of vector spaces

$$\begin{array}{rcl} \tau : \mathbb{k}[y_{1,1}, \dots, y_{1,\gamma_1}, \dots, y_{e,\gamma_e}] & \to & R_+ = \mathbb{k}[Y_{1,1}, \dots, Y_{1,\gamma_1}, \dots, Y_{e,\gamma_e}] \\ y_{c,j} & \mapsto & 1 - q^{-c} Y_{c,j} \end{array}$$

induces an isomorphism of vector spaces

$$\begin{array}{rcl} \tau_{\mathbf{u}_K} : \widehat{\Lambda}_{\mathbf{u}_K} & \to & \widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}^{\mathbf{u}_K} = \hat{R}_{+,\mathbf{u}_K} \overline{\mathbf{v}}^{(K)} \\ f\overline{\mathbf{v}}^{(K)} & \mapsto & \tau(f)\overline{\mathbf{v}}^{(K)} \end{array}$$

and thus a total isomorphism

$$\boldsymbol{\tau} = \bigoplus_{\substack{K \subseteq \mathbb{I} \\ \mathbf{u}_K \in \mathbf{U}_K}} \tau_{\mathbf{u}_K} : \qquad \widehat{\Lambda} \to \widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}. \tag{9.1}$$

From now on we will identify these two vector spaces via our chosen isomorphism. With this identification we can compare our endomorphism algebras:

Proposition 9.1. *The isomorphism $\boldsymbol{\tau}$ can be extended to an algebra isomorphism $\boldsymbol{\tau} : \widehat{\mathbf{C}}_{\mathbf{i}} \to \widehat{\mathcal{S}}_{\mathbf{i}}$ which*

- *identifies the subalgebra of $\widehat{\mathbf{C}}_{\mathbf{i}}$ generated by all $e(\mathbf{u}_J)pe(\mathbf{u}_J)$ for all $J \subset \mathbb{I}, \mathbf{u}_J \in \mathbf{U}_J$ and $p \in \Lambda_{\mathbf{u}_J}$ with the algebra $\widehat{\mathcal{Q}}_{\mathbf{i}}$ from Section 5.6,*
- *identifies splits in the sense that, for any $J \subset K \subseteq$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, it maps $\curlyvee_{\mathbf{u}_K}^{\mathbf{u}_J}$ to $\overset{+}{e}_{\mathbf{u}_J} \mathbf{b}_{J,K}^1 \overset{+}{e}_{\mathbf{u}_K}$,*
- *identifies merges in the sense that, in case $J \subset K$, $|J| = |K| - 1$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, the generator $\curlywedge_{\mathbf{u}_J}^{\mathbf{u}_K}$ maps to $\overset{+}{e}_{\mathbf{u}_K} \mathbf{b}_{K,J}^1 \overset{+}{e}_{\mathbf{u}_J} P^{-1}$ for an invertible power series P.*

From our identification of local and global indices in (5.10), it is immediate that $\boldsymbol{\tau}$ extends the isomorphism (7.2) given in Section 7.2.

Proof. It follows immediately from Proposition 5.18 and Lemma 4.19 that the action of the algebra $\widehat{\mathcal{Q}}_{\mathbf{i}}$ coincides with the action of the subalgebra of $\widehat{\mathbf{C}}_{\mathbf{i}}$ generated by all $e(\mathbf{u}_J)pe(\mathbf{u}_J)$ under the identification $\boldsymbol{\tau}^{-1}$. Hence the first claim holds. It is also clear from Example 4.5 that for $J \subset K$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, the action of $\overset{+}{e}_{\mathbf{u}_J} \mathbf{b}^1_{J,K} \overset{+}{e}_{\mathbf{u}_K}$ on $\widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}$ translates directly to the action of $\curlyvee^{\mathbf{u}_J}_{\mathbf{u}_K}$ under $\boldsymbol{\tau}^{-1}$. Hence the second assertion holds as well. We now claim that for $J \subset K$, $|J| = |K| - 1$ and $\mathbf{u}_K = (\mathbf{u}', K) \in \mathbf{U}_K$ a simple merge of $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, the action of $\overset{+}{e}_{\mathbf{u}_K} \mathbf{b}^1_{K,J} \overset{+}{e}_{\mathbf{u}_J}$ on $\widehat{\overline{\mathbb{P}}(\mathcal{S})}_{\mathbf{i}}$ translates into the action of $\curlywedge^{\mathbf{u}_K}_{\mathbf{u}_J} P$ for an invertible power series P. Again, to ease terminology, we check this in the case of $K = \mathbb{I}$ and $\mathbf{u}_J = (1^{a_1}, 2^{a_2}, \cdots, e^{a_e} | 1^{b_1}, 2^{b_2}, \cdots, e^{b_e})$. Since the calculations are local this is sufficient. Recall from Proposition 6.11 that for $f \in \hat{R}_{+,\mathbf{u}_J}$

$$e_{\mathbf{u}'} \widehat{\overline{\rho}}(\mathbf{b}^1_{K,J}) f \overline{\mathbf{v}}^{(J)} = e_{\mathbf{u}'} \pitchfork^{d_1, \ \dots \ d_2, \ \dots \ d_n}_{a_1, b_1, a_2, b_2, \dots, a_e, b_e} \sigma^{\mathbf{u}'_K}_{\mathbf{u}_J} \prod_{k=a+1}^{n} \prod_{j=1}^{a} \theta_{j,k} f \overline{\mathbf{v}}^{(K)}.$$

Translating this into the variables $Y_{i,j}$, notice that $\sigma^{\mathbf{u}_K}_{\mathbf{u}_J}$ becomes superfluous as it is precisely the element mapping $(i,j)_{\mathbf{u}_J}$ from Definition 5.12 to $(i,j)_{\mathbf{u}'_K}$ and is hence the identity on the variable $Y_{i,j}$. We obtain

$$e_{\mathbf{u}'} \widehat{\overline{\rho}}(\mathbf{b}^1_{K,J}) f \overline{\mathbf{v}}^{(J)} = e_{\mathbf{u}'} \pitchfork^{d_1, \ \dots \ d_2, \dots, d_e}_{a_1, b_1, a_2, b_2, \dots, a_e, b_e} \prod_{i=1}^{e} \prod_{s=1}^{e} \prod_{j=1}^{a_i} \prod_{k=a_s+1}^{d_s} \frac{qY_{i,j} - Y_{s,k}}{Y_{i,j} - Y_{s,k}}$$

Under $\boldsymbol{\tau}^{-1}$, multiplication by $\prod_{i=1}^{e} \prod_{s=1}^{e} \prod_{j=1}^{a_i} \prod_{k=a_s+1}^{d_s} \frac{qY_{i,j} - Y_{s,k}}{Y_{i,j} - Y_{s,k}}$ translates to multiplication by

$$\begin{aligned}
& \prod_{i=1}^{e} \prod_{s=1}^{e} \prod_{j=1}^{a_i} \prod_{k=a_s+1}^{d_s} \frac{q^{i+1}(1-y_{i,j}) - q^s(1-y_{s,k})}{q^i(1-y_{i,j}) - q^s(1-y_{s,k})} \\
= & \prod_{i=1}^{e} \prod_{j=1}^{a_i} \left(\prod_{k=a_i+1}^{d_i} \frac{1-q+qy_{i,j} - y_{i,k}}{y_{i,j} - y_{i,k}} \prod_{k=a_{i+1}+1}^{d_{i+1}} \frac{-q(y_{i,j} - y_{i+1,k})}{1-q-y_{i,j}+qy_{i+1,k}} \mathbf{R} \right) \\
= & \left(\prod_{i=1}^{e} \prod_{j=1}^{a_i} \frac{\prod_{k=a_{i+1}+1}^{d_{i+1}} y_{i,j} - y_{i+1,k}}{\prod_{k=a_i+1}^{d_i} y_{i,j} - y_{i,k}} \right) \mathbf{P}.
\end{aligned}$$

where we abbreviated

$$\begin{aligned}
\mathbf{R} &= \prod_{\substack{s=1\\ s\neq i,i+1}}^{e} \prod_{k=a_s+1}^{d_s} \frac{q^{i+1}(1-y_{i,j})-q^s(1-y_{s,k})}{q^i(1-y_{i,j})-q^s(1-y_{s,k})}\\
\mathbf{P} &= \prod_{i=1}^{e}\prod_{j=1}^{a_i}\Big(\prod_{k=a_i+1}^{d_i}(1-q+qy_{i,j}-y_{i,k})\prod_{k=a_{i+1}+1}^{d_{i+1}}\frac{-q}{1-q-y_{i,j}+qy_{i+1,k}}\mathbf{R}\Big).
\end{aligned}$$

Note that $\mathbf{P}$ is an invertible power series in $\Lambda_{\mathbf{u}_K}$. Hence the third assertion holds as well. By definition of the modified quiver Schur algebra we have mapped all generators to the corresponding elements in $\widehat{\mathcal{S}}_{\mathbf{i}}$ by identifying their action on the faithful representations. This implies that $\boldsymbol{\tau}$ is injective. By Proposition 6.19, the image of $\boldsymbol{\tau}$ contains a generating set for $\widehat{\mathcal{S}}_{\mathbf{i}}$, so $\boldsymbol{\tau}$ is also surjective and hence an isomorphism. □

9.2. **The isomorphism $\mathbf{B_i} \cong \mathbf{A_i}$ of (shifted) quiver Schur algebras.** We next show that the shifted quiver Schur algebra is isomorphic to the ordinary quiver Schur algebra. We start with some preparation. First, we again identify the vector spaces underlying the faithful representations.

For $\mathbf{u}_J \in \mathbf{U}_J$, set ${}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_J} := E_{\mathbf{u}_J}\mathring{\Lambda}_{\mathbf{u}_J}$ and ${}^{\downarrow}\mathring{\Lambda} = \bigoplus_{\substack{J\subseteq\mathbb{I}\\ \mathbf{u}_J\in\mathbf{U}_J}} {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_J}$. Fix the vector space isomorphism

$$\begin{aligned}
\kappa_{\mathbf{u}_J} \;:\; \mathring{\Lambda}_{\mathbf{u}_J} &\to {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_J}\\
f &\mapsto E_{\mathbf{u}_J} f
\end{aligned}$$

and the induced vector space isomorphism

$$\kappa = \bigoplus_{\substack{J\subseteq\mathbb{I}\\ \mathbf{u}_J\in\mathbf{U}_J}} \kappa_{\mathbf{u}_J} : \mathring{\Lambda} \to {}^{\downarrow}\mathring{\Lambda}.$$

Lemma 9.2. *Endowing ${}^{\downarrow}\mathring{\Lambda}$ with a representation of $\mathbf{B_i}$ via κ, the induced action is given by the same formulae as the action of $\mathbf{A_i}$ on $\mathring{\Lambda}$, i.e.*

- *the idempotents $e(\mathbf{u}_J)$, for $\mathbf{u}_J \in \mathbf{U}_J$ for any J, act as the projection onto the summand ${}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_J}$,*
- *the polynomials $e(\mathbf{u}_J)pe(\mathbf{u}_J)$, for any J and $p \in {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_J}$, act as multiplication by p on the summand ${}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_J}$.*
- *the split $\vec{\curlyvee}_{\mathbf{u}_K}^{\mathbf{u}_J}$ for any $J \subset K$ and $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, acts on an element f with $f \in {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}'_{K'}}$ by*

$$f \;\mapsto \begin{cases} \mathring{\mathrm{E}}_{\mathbf{u}_J}^{\mathbf{u}_K} f \in {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_J} & \text{if } \mathbf{u}'_{K'} = \mathbf{u}_K,\\ 0 & \text{otherwise.}\end{cases}$$

- *the merge* $\curlywedge_{\mathbf{u}_J}^{\vec{\mathbf{u}}_K}$ *for any* $J \subset K$ *and* $\mathbf{u}_J = (\mathbf{u}, J) \in \mathbf{U}_J$, *act on an element* $f \in \mathring{\Lambda}_{\mathbf{u}'_{J'}}$ *by*

$$f \quad \mapsto \begin{cases} \mathring{\Delta}(f) \in {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_K} & \textit{if } \mathbf{u}'_{J'} = \mathbf{u}_J, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The actions of idempotents and polynomials are immediate. In order to compare the actions of splits, consider the diagram

$$\begin{tikzcd} \mathring{\Lambda}_{\mathbf{u}_K} \arrow[r, "\cdot \mathring{\mathrm{E}}_{\mathbf{u}_K}"] \arrow[d, "1"] & {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_K} \arrow[d] \\ \mathring{\Lambda}_{\mathbf{u}_J} \arrow[r, "\cdot \mathring{\mathrm{E}}_{\mathbf{u}_J}"] & {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_J} \end{tikzcd}$$

Then the third claim of the lemma is equivalent to the commutativity of the required diagram, which is equivalent to the fact that the induced action of $\curlyvee_{\mathbf{u}_K}^{\vec{\mathbf{u}}_J}$ on ${}^{\downarrow}\mathring{\Lambda}$ is indeed given by multiplication with $\mathring{\mathrm{E}}_{\mathbf{u}_J}/\mathring{\mathrm{E}}_{\mathbf{u}_K} = \mathring{\mathrm{E}}_{\mathbf{u}_J}^{\mathbf{u}_K}$.

For a merge $\curlywedge_{\mathbf{u}_J}^{\vec{\mathbf{u}}_K}$, consider the diagram

$$\begin{tikzcd} \mathring{\Lambda}_{\mathbf{u}_K} \arrow[r, "\cdot \mathring{\mathrm{E}}_{\mathbf{u}_K}"] & {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_K} \\ \mathring{\Lambda}_{\mathbf{u}_J} \arrow[r, "\cdot \mathring{\mathrm{E}}_{\mathbf{u}_J}"] \arrow[u, "\mathring{\Delta}(\mathring{\mathrm{E}}_{\mathbf{u}_J}^{\mathbf{u}_K} -)"] & {}^{\downarrow}\mathring{\Lambda}_{\mathbf{u}_J}. \arrow[u] \end{tikzcd}$$

Again we need to verify commutativity, which again stems from the fact that the action of $\curlywedge_{\mathbf{u}_J}^{\vec{\mathbf{u}}_K}$ on ${}^{\downarrow}\mathring{\Lambda}$ is given by

$$f \mapsto \mathring{\mathrm{E}}_{\mathbf{u}_K} \mathring{\Delta}(\mathring{\mathrm{E}}_{\mathbf{u}_J}^{\mathbf{u}_K} \mathring{\mathrm{E}}_{\mathbf{u}_J}^{-1} f) = \mathring{\mathrm{E}}_{\mathbf{u}_K} \mathring{\Delta}(\mathring{\mathrm{E}}_{\mathbf{u}_K}^{-1} f).$$

Noticing that $\mathring{\mathrm{E}}_{\mathbf{u}_K}$ commutes with $\pitchfork_{I_{\mathbf{u}_J}}^{I_{\mathbf{u}_K}}$ by $\mathfrak{S}_{\mathbf{u}_K}$-invariance of $\mathring{\mathrm{E}}_{\mathbf{u}_K}$, the claim follows. ☐

Lemma 9.3. *The representation of* $\mathbf{B_i}$ *on* ${}^{\downarrow}\mathring{\Lambda}$ *is faithful. Moreover, the action of* $\mathbf{A_i}$ *restricted to* ${}^{\downarrow}\mathring{\Lambda}$ *is equal to the action of* $\mathbf{B_i}$.

Proof. Directly from the proof of the Lemma 9.2, we see that the action of $\mathbf{A_i}$ on $\mathring{\Lambda}$ when restricted to ${}^{\downarrow}\mathring{\Lambda}$ is equal to the action of $\mathbf{B_i}$. Since the representation of $\mathbf{A_i}$ on $\mathring{\Lambda}$ was faithful, the representation of $\mathbf{B_i}$ on ${}^{\downarrow}\mathring{\Lambda}$ is faithful as well. The second statement follows from the commutative diagrams above. ☐

We claim that the canonical embedding $\iota : {}^{\downarrow}\mathring{\Lambda} \hookrightarrow \mathring{\Lambda} \;\;:\;\; f \mapsto f$ induces an isomorphism of algebras as follows.

Proposition 9.4. *The algebras* $\mathbf{A_i}$ *and* $\mathbf{B_i}$ *are isomorphic.*

Proof. Since the action of $\mathbf{A_i}$ restricted to ${}^{\downarrow}\mathring{\Lambda}$ is equal to the action of $\mathbf{B_i}$ by Lemma 9.3, we see that ${}^{\downarrow}\mathring{\Lambda}$ is a faithful subrepresentation of $\mathring{\Lambda}$ for $\mathbf{A_i}$. The algebra $\mathbf{A_i}$ is therefore completely defined by its action on ${}^{\downarrow}\mathring{\Lambda}$, and we obtain the desired isomorphism $\mathbf{A_i} \cong \mathbf{B_i}$ from Lemma 9.2. □

9.3. **The isomorphism $\mathbf{C_i} \cong \mathbf{B_i}$.** In order to prove that $\widehat{\mathcal{S}}_{\mathbf{i}}$ and $\mathbf{A_i}$ are isomorphic, it now remains to show that $\mathbf{C_i}$ and $\mathbf{B_i}$ are isomorphic.

In order to do this, we define a bijection

$$\begin{array}{rcrcl} \sim & : & \bigcup_{J\subset\mathbb{I}} \mathbf{U}_J & \to & \bigcup_{J\subset\mathbb{I}} \mathbf{U}_J : \\ & & \mathbf{u}_J & \mapsto & \widetilde{\mathbf{u}_J}. \end{array}$$

where for $\mathbf{u}_J = (u_1, \cdots, u_{t_1} | u_{t_1+1}, \cdots, u_{t_2} | \cdots | u_{t_{r-1}+1}, \cdots, u_{t_r})$, we set

$$\widetilde{\mathbf{u}_J} = (u_{t_{r-1}+1}, \cdots, u_{t_r} | u_{t_{r-2}+1}, \cdots, u_{t_{r-1}} | \cdots | u_1, \cdots, u_{t_1}).$$

We further define the inner automorphism of $\mathfrak{S}_n$ which is given by conjugation with the longest element w_0^J of $D^{\mathbb{I}}_{\emptyset,J}$ and notice that this interchanges $\mathfrak{S}_{\mathbf{u}_J}$ and $\mathfrak{S}_{\widetilde{\mathbf{u}_J}}$. It induces the isomorphism θ of vector spaces

$$\begin{array}{ccc} \Bbbk[y_{1,1}, \dots, y_{1,d_1}, y_{2,1}, \dots, y_{2,d_2}, \dots, y_{e,d_e}] & & y_{c,j} \\ \Big\downarrow \theta & & \Big\downarrow \\ \Bbbk[z_{1,1}, \dots, z_{1,d_1}, z_{2,1}, \dots, z_{2,d_2}, \dots, z_{e,d_e}] & & z_{c,w_0^J(j)} = -z_{c,d_c+1-j} \end{array}$$

which restricts to an isomorphism θ of vector spaces

$$\theta : \quad \Lambda_{\mathbf{u}_J} \ \overset{\sim}{\to} \ \mathring{\Lambda}_{\widetilde{\mathbf{u}_J}}.$$

Example 9.5. Consider for instance $\mathbf{u}_J = (1,1,1,1,2,2,3|1,2,3|1,1,2,3,3)$, then we have $\widetilde{\mathbf{u}_J} = (1,1,2,3,3,|1,2,3|1,1,1,1,2,2,3)$ and $\mathrm{W}_J \cong \mathfrak{S}_7 \times \mathfrak{S}_3 \times \mathfrak{S}_4$ and

$$\mathfrak{S}_{\mathbf{u}_J} \cong (\mathfrak{S}_4 \times \mathfrak{S}_2 \times \{1\}) \times (\{1\} \times \{1\} \times \{1\}) \times (\mathfrak{S}_2 \times \{1\} \times \mathfrak{S}_2).$$

Under conjugation by w_0^J this is sent to

$$(\mathfrak{S}_2 \times \{1\} \times \mathfrak{S}_2) \times (\{1\} \times \{1\} \times \{1\}) \times (\mathfrak{S}_4 \times \mathfrak{S}_2 \times \{1\}) \cong \mathfrak{S}_{\widetilde{\mathbf{u}_J}}.$$

Proposition 9.6. *There is an isomorphism of algebras*

$$\begin{array}{rcl} \mathbf{C_i} & \longrightarrow & \mathbf{B_i} \\ e(\mathbf{u}_J) & \longmapsto & e(\widetilde{\mathbf{u}_J}) \\ pe(\mathbf{u}_J) & \longmapsto & \theta(p)e(\widetilde{\mathbf{u}_J}) \\ \curlyvee^{\mathbf{u}_J}_{\mathbf{u}_K} & \longmapsto & \overset{\rightarrow}{\curlyvee}^{\widetilde{\mathbf{u}_J}}_{\widetilde{\mathbf{u}_K}} \\ \curlywedge^{\mathbf{u}_K}_{\mathbf{u}_J} & \longmapsto & \overset{\rightarrow}{\curlywedge}^{\widetilde{\mathbf{u}_K}}_{\widetilde{\mathbf{u}_J}}. \end{array}$$

Proof. We check that the isomorphism $\theta : \Lambda \to \mathring{\Lambda}$ intertwines the actions of $\mathbf{C_i}$ and $\mathbf{B_i}$ with respect to the isomorphism given in the proposition. It is obvious that this is true for the idempotents and the polynomials, as well as the splits. Looking at the action of merges, the fact that conjugation with w_0^J sends $\pitchfork_{I_{\mathbf{u}_J}}^{I_{\mathbf{u}_K}}$ to $\pitchfork_{I_{\widetilde{\mathbf{u}_J}}}^{I_{\widetilde{\mathbf{u}_K}}}$, we see that, in order to prove the proposition, it suffices to show that

$$\theta(\mathtt{E}_{\mathbf{u}_J}) = \mathring{\mathtt{E}}_{\widetilde{\mathbf{u}_J}} \quad \text{and} \quad \theta(\mathtt{S}_{\mathbf{u}_J}) = \mathring{\mathtt{S}}_{\widetilde{\mathbf{u}_J}}.$$

Now notice that the term $y_{i,j} - y_{i+1,k}$ appears in $\mathtt{E}_{\mathbf{u}_J}$ (and thus the term $z_{i+1,d_{i+1}+1-k} - z_{i,d_i+1-j}$ appears in $\theta(\mathtt{E}_{\mathbf{u}_J})$) if and only if the jth appearance of i in $\mathbf{u}_J$ is in an earlier segment than the kth appearance of $i+1$. Since applying $\tilde{}$ to $\mathbf{u}_J$ reverses segments, this is equivalent to the (d_i+1-j)th appearance of i in $\widetilde{\mathbf{u}_J}$ being in a later segment than the $(d_{i+1}+1-k)$th appearance of i. The latter is equivalent to the term $z_{i+1,d_{i+1}+1-k} - z_{i,d_i+1-j}$ appearing in $\mathring{\mathtt{E}}_{\widetilde{\mathbf{u}_J}}$. The claim that $\theta(\mathtt{S}_{\mathbf{u}_J}) = \mathring{\mathtt{S}}_{\widetilde{\mathbf{u}_J}}$ is checked analogously. □

9.4. **The main theorem.** We are now prepared to prove our main result:

Theorem 9.7 (Isomorphism Theorem)**.** *There is an isomorphism of algebras*

$$\widehat{\mathcal{S}}_{\mathbf{i}} \;\cong\; \widehat{\mathbf{A}}_{\mathbf{i}}.$$

Via this isomorphism $\widehat{\mathbf{A}}_{\mathbf{i}}$ inherits a grading from $\widehat{\mathcal{S}}_{\mathbf{i}}$.

Proof. Composing the isomorphism from Proposition 9.1 with the completions of the respective isomorphisms in Propositions 9.6 and 9.4 provides the required isomorphism. In formulae, the composition

$$\widehat{\mathcal{S}}_{\mathbf{i}} \xrightarrow{\text{Proposition 9.1}} \widehat{\mathbf{C}}_{\mathbf{i}} \xrightarrow{\text{Proposition 9.6}} \widehat{\mathbf{B}}_{\mathbf{i}} \xrightarrow{\text{Proposition 9.4}} \widehat{\mathbf{A}}_{\mathbf{i}}.$$

is an isomorphism of algebras. □

10. The example $\mathrm{GL}_2(\mathbb{Q}_5)$ in characteristic 3

We finish with an explicit example, namely we consider the unipotent block $\mathcal{B}$ (that is the block containing the trivial representation) of the category of smooth representations for $\mathrm{GL}_2(\mathbb{Q}_5)$ over an algebraically closed field $\Bbbk$ of characteristic 3, so $e = 2$.

10.1. **The (completed) quiver Schur algebra.** Let $\mathcal{B}^1$ as in (1.1) and $\mathcal{B}^1_{\mathbf{a}}$ the full subcategory of $\mathcal{B}^1$ consisting of representations with generalized central character $\chi_{\mathbf{a}}$ where $\mathbf{a} = (q, q^2)$. Equivalently, $\mathcal{B}^1_{\mathbf{a}}$ is the full subcategory of $\mathcal{S}-\mathrm{Mod}$ of all representations with generalized central character $\chi_{\mathbf{a}}$.

Theorem 10.1. *Let $n = 2 = e$. Then the quiver Schur algebra $\mathbf{A_i}$ for $\mathbf{i} = (1,2)$ is (as graded algebra) isomorphic to the path algebra B of the following quiver*

$$
\begin{tikzcd}[column sep=huge]
(1|2) \arrow[loop above, "x_{1,1}e(1|2)"] \arrow[loop below, "x_{2,1}e(1|2)"'] \arrow[r, bend left, "\curlywedge^{(1,2)}_{(1|2)}"] & (1,2) \arrow[loop above, "x_{1,1}e(1,2)"] \arrow[loop below, "x_{2,1}e(1,2)"'] \arrow[l, bend left, "\curlyvee^{(1|2)}_{(1,2)}"] \arrow[r, bend left, "\curlyvee^{(2|1)}_{(1,2)}"] & (2|1) \arrow[loop above, "x_{1,1}e(2|1)"] \arrow[loop below, "x_{2,1}e(2|1)"'] \arrow[l, bend left, "\curlywedge^{(1,2)}_{(2|1)}"]
\end{tikzcd}
\tag{10.1}
$$

with grading given by putting the horizontal arrows in degree 1 and the loops in degree 2 modulo the (homogeneous) relations

$$
\begin{aligned}
\overset{(1|2)}{\underset{(12)}{\curlyvee}}\ \overset{(12)}{\underset{(1|2)}{\curlywedge}} &= (x_{2,1}-x_{1,1})e((1|2)),\\
\overset{(2|1)}{\underset{(12)}{\curlyvee}}\ \overset{(12)}{\underset{(2|1)}{\curlywedge}} &= (x_{1,1}-x_{2,1})e((2|1)),\\
\overset{(12)}{\underset{(1|2)}{\curlywedge}}\ \overset{(1|2)}{\underset{(12)}{\curlyvee}} &= -\ \overset{(12)}{\underset{(2|1)}{\curlywedge}}\ \overset{(2|1)}{\underset{(12)}{\curlyvee}} \ = \ (x_{1,1}-x_{2,1})e((1,2)),\\
x_{i,1}\overset{\mathbf{u}_J}{\underset{(12)}{\curlyvee}} &= \overset{\mathbf{u}_J}{\underset{(12)}{\curlyvee}}x_{i,1} \quad \textit{for } i\in\{1,2\}, \mathbf{u}_J\in\{(1|2),(2|1)\},\\
x_{i,1}\overset{(12)}{\underset{\mathbf{u}_J}{\curlywedge}} &= \overset{(12)}{\underset{\mathbf{u}_J}{\curlywedge}}x_{i,1} \quad \textit{for } i\in\{1,2\}, \mathbf{u}_J\in\{(1|2),(2|1)\}.
\end{aligned}
$$

(Recall that the path algebra of this quiver is the $\Bbbk$-algebra with basis all possible paths obtained by concatenating the arrows, including the three paths of length zero corresponding to the vertices of the graph. The multiplication of two paths is the path obtained by concatenation if this makes sense and zero otherwise.)

Remark 10.2. Note that, since the algebra is non-negatively graded and 3-dimensional in degree zero, the three idempotents $e(1|2)$, $e(2|1)$, and $e(1,2)$ must be primitive.

Proof. By Remark 10.2, the given three idempotent are primitive and by definition pairwise orthogonal, hence the quiver has three vertices. The idempotents together with the elements corresponding to the arrows generate the quiver Schur algebra by Definition 8.8. The relations are easily verified on the faithful representation from Definition 8.8. The fact that these are all the relations can be checked again by a direct calculation, or follows from the basis theorem in [SW11]. $\square$

Remark 10.3. Note that the elements $e(1|2)$, $e(2|1)$, $x_{i,1}e(1|2)$, $x_{i,1}e(1|2)$ with $i = 1, 2$ together with $\between^{(2|1)}_{(1|2)} = \curlyvee^{(2|1)}_{(1,2)} \curlywedge^{(1,2)}_{(1|2)}$ and $\between^{(1|2)}_{(2|1)} = \curlyvee^{(1|2)}_{(1,2)} \curlywedge^{(1,2)}_{(2|1)}$ generate a graded subalgebra of $B \cong \mathbf{A_i}$ isomorphic to the quiver Hecke or KLR algebra attached in in [KL09] and [Rou08] to the cyclic quiver and the sequence $\mathbf{i} = (1, 2)$.

From our main theorem we get the following consequence.

Corollary 10.4. *Let* $\mathrm{G} = \mathrm{GL}_2(\mathbb{Q}_5)$ *and assume* $\ell = 3$, *hence* $e = 2$. *Then the category* $\mathcal{B}^1_{\mathbf{a}}$ *of representations in* $\mathcal{B}^1$ *with generalised central character* $\chi_{\mathbf{a}}$ *is equivalent to the category of* $\hat{B}$*-modules, where* $\hat{B}$ *is the completion of* B *at the maximal ideal* $(x_{1,1}, x_{2,1})$ *of* $\Bbbk[x_{1,1}, x_{2,1}] \subset B$.

Proof. By Theorem 9.7 and Theorem 10.1, $\hat{B}$ is isomorphic to the completed affine Schur algebra from Proposition 5.1. Hence it is isomorphic to the completion of the endomorphism ring of a projective progenerator of $\mathcal{B}^1$ by (1.1), the module category over which gives precisely the category of objects in $\mathcal{B}^1$ with the given generalised central character. $\square$

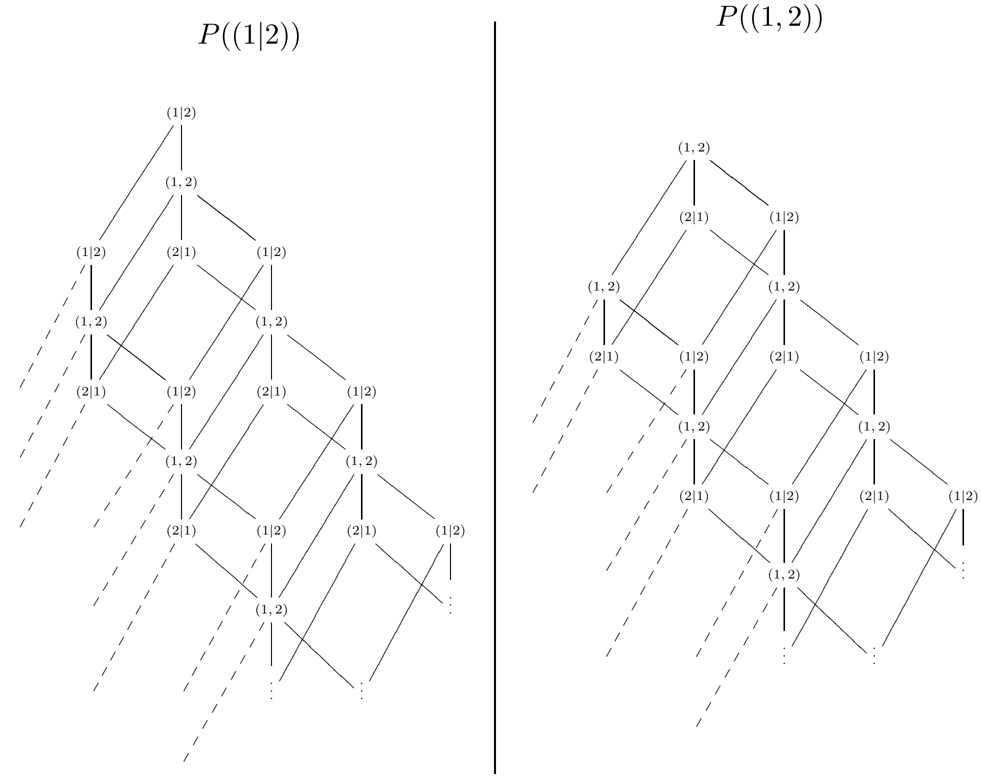
Remark 10.5. Since every irreducible representation in $\mathcal{B}$ is smooth and therefore admissible (see e.g. [BI11, Theorem 4.42] or [BZ76, Theorem 3.25]), it has a central character by Schur's Lemma. The category of objects in $\mathcal{B}^1$ with some generalised central character thus includes blocks of finite length objects in $\mathcal{B}^1$.

Note that $\mathrm{End}_B(Be) \cong \Bbbk[x_1, x_2]$, generated by $ex_{1,1}e$ and $ex_{1,2}e$ for any $e \in \{e(1|2), e(2|1), e(12)\}$. Moreover, $\mathrm{Hom}_B(Be, Be') \cong \Bbbk[x_1, x_2]$ as vector spaces for any pair (e, e') of these idempotents. It is free as a left $\mathrm{End}_B(Be)$-module and as a right $\mathrm{End}_B(Be')$-module of rank 1 with basis the minimal degree morphism in $\mathrm{Hom}_B(Be, Be')$. Hence B can be viewed as a $\Bbbk[x_1, x_2]$-algebra. As such it is quadratic, that is, generated in degree one (by the morphisms corresponding to the arrows given by simple merges and splits) with relations in degree two.

10.2. **Indecomposable projectives.**

The indecomposable projective B-modules $P((1|2))$, $P((2|1))$ and $P((1,2))$ look as in the pictures below, where the numbers stand for the corresponding simple object and the lines for a basis vector in Ext^1.

The part indicated by the non-dashed lines should be extended to infinity at the bottom and then the whole resulting part is copied infinitely many times (indicated by the dashed lines), once for each power of $(x_{1,1} + x_{2,1})$. The structure of $P((2|1))$ is similar to that of $P((1|2))$, just with $(1|2)$ and $(2|1)$ swapped.



10.3. **The corresponding irreducible representations.** The labelling of the primitive idempotents in (10.1) corresponds to a labelling of the three simple modules in $\mathcal{B}$. Explicitly, we have

- $(1|2)$ (corresponding to the trivial representation),
- $(2|1)$ (corresponding to the composition of the valuation on $\mathbb{Q}_5$ and the determinant), and
- $(1,2)$ (corresponding to the cuspidal representation).

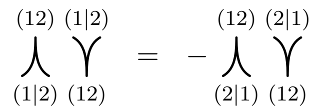
To verify this, note that the first two idempotents are contained in the quiver Hecke algebra (see Remark 10.3), hence correspond to the two simple representations which are not cuspidal. For these two the identification is a matter of conventions.

10.4. **The Extquiver of** B**.**

Corollary 10.6. *In the situation from above, the Ext-quiver of* B *is*

$$\begin{tikzcd}
(1|2) \arrow[loop above, "ze(1|2)"] \arrow[r, bend left, "\curlywedge^{(1,2)}_{(1|2)}"] & (1,2) \arrow[loop above, "ze(1,2)"] \arrow[l, bend left, "\curlyvee^{(1|2)}_{(1,2)}"] \arrow[r, bend left, "\curlyvee^{(2|1)}_{(1,2)}"] & (2|1) \arrow[loop above, "ze(2|1)"] \arrow[l, bend left, "\curlywedge^{(1,2)}_{(2|1)}"]
\end{tikzcd} \tag{10.2}$$

and the relations are that $z = ze(1|2) + ze(2|1) + ze(1,2)$ *is central and*



Proof. This follows directly from the theorem by setting $z = x_{1,1} + x_{2,1}$. $\square$

In this example one can in fact verify our general expectation that $\mathcal{B}$ only differs from $\mathcal{B}^1$ by self-extensions of the simple cuspidal representation, and thus $\mathcal{B}^1$ contains more or less all information about the unipotent block $\mathcal{B}$.

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